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EVOLUTION OF A "SCIENCE OF MANAGING" IN AMERICA*

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I

Wherever people have gathered to pursue a common and desired end, there has been an inevitable necessity to organize minds, hands, materials, and the use of time for efficient and contributive work. Man has learned that individual and personal rewards derive largely from an harmonious combination of individual work and teamwork in a soundly organized frame of reference, and thus the core of the history of "Scientific Management" is formed from his search for the techniques of joint but voluntary participation while still preserving individual initiative, creative imagination, and increasingly productive output.

Any historical survey, to be of more than passing interest, and to be more than a simple chronology of dates and names, needs to seek out the philosophical drives which both stimulated and limited the progress of such a scientific approach to more rational conception and performance of managerial work, that is, of securing results through the organized efforts of others.

While it is important to know when things happened, this knowledge only becomes significant and useable when it is understood *why* things happened. This paper, therefore, is intended to outline the gradual historical development of the search for the basic principles of a "Science of Managing," and thus of Scientific Management, rather than to be a simple recounting of experiment, methodology, or significant writings in that field.

II

The philosophical sciences, and that dealing with the work of Managing is one of them, deal with concepts and abstractions not *easily* tested or proved quantitatively, especially on a current basis, and not *readily* subject to exact

* The views expressed herein do not necessarily represent any policy of the Institute of Management Sciences. Other articles on allied subjects will appear in this JOURNAL from time to time.

delineation, definition, standardization, or measurement. The ability to observe, to classify, to synthesize, and to act, are at least partially inhibited by the relationship and reciprocal impact of the observer and his environment.

The observer deals with the classic study of man, and he deals with one of its most complex branches, the seeking of orderly process out of divergent wills, unclear or even contradictory objectives, and transient and often unmeasurable forces. Because each observer is an integral factor in his own study, complete objectivity is impossible, and he is often faced with the conflict of that which reason tells him must be so and that which emotion tells him should be so.

But, by acceptance of the limitations in method, rather than rejection because of the difficulties, cumulative progress has been consistently sought and progressively achieved. As a result, a "Management Movement," which lived only in the hopes of a relatively few dedicated volunteers as recently as only three decades ago, has, as of today, grown irresistably to an organized and intensifying worldwide drive and to an increasingly international educational force.

Despite such visible and accelerating progress in so short an historical era, the Science of Managing, when compared to the physical sciences, may still seem slow in evolution and uncertain in direction. Yet, fairly judged, it is neither.

The apparent slowness has only reflected the normal systemic inertia encountered wherever tradition and prejudice guide men's actions and wherever there is deliberate observation of cause and effect. Uncertainty and delay in such progress has really been the result neither of any lack of a true goal nor of unwillingness to follow a just direction, but is, rather, the result of a need to consider every avenue of possible study and, by the very nature of things, to reject far more than can be accepted.

What is being sought? Certainly not the solution of isolated problems or the rectification of each pressing business or social exigency. That would be in the nature of art. Certainly not merely the kind of "progress" determined by ability to answer the simple questions: "Will it work?" "Can you get by with it?"

What has been, and still is, sought in all these recurring efforts to define and develop a true "Science of Managing" are, rather, those kinds of fundamental principles which are the essential scientific foundation of all generalization based on classified observations and which give meaning, accuracy, and dependability to formulation of rules of action or policies, that, given a particular set of conditions, can be used or applied as guides with confidence in their effectiveness.

In other words, the search of the organized Management Movement and of the growing thousands of individual participants, both scholars and practitioners, is for principles, distilled out of what is now increasingly widely recorded experience in the social sciences, which can be applied to any and all situations involving the demands of Leadership by reason and persuasion rather than merely by rank or dictation.

To consider the shop, the office, or even the total business as something apart and unique from the more general problem of *organizing for accomplishment*, is, therefore, to take the short and unproductive view. The stubborn refusal of

disciples of scientific management to fall into that kind of mental trap is itself one of the firm reasons for the growing understanding and acceptance—locally, nationally, and internationally—that management principles can be applied to guidance of all kinds of organized efforts and can be a base for Management Education which leads to professional rather than whimsical managerial responsibility.

How is this "search" being conducted? Essentially, it has gone forward on two levels. First, there has been the elemental, "action" group whose members will always be necessary to avoid crises, but who rarely have time for rigorous thought. They have been putting out fires rather than developing methods of fire prevention. At best, they keep the work of the day from lagging, and they provide data for the file of experience.

Second, and most important, are those who, by time and circumstance, are determined to find and follow carefully measured paths, distinct though not remote from immediate demands.

Fortunately, this latter group is increasing both in size and influence, and in recent years their work has been credited with growth from deep roots and not merely from a knowing wave of the hand. It is because such men, and women, have boldly and persuasively demonstrated that there are basic truths which underlie the study of the relationships of men in common effort, and have shown that these truths can be demonstrated, applied, and measured, that acceptance of the needed fundamental study to develop an accepted "Science of Managing" is being won.

There are still many who regard the idea and concept of a "Science of Managing" as a scholarly diversion, a type of *post hoc* rationalization that is created and dies in the confines of the lecture hall. But, especially in the United States, while there have been, to be sure, the theorists and while their contributions have been indispensable, there have also been the practical men, and more often than not, both theoretical concepts and practical applications have found common spokesmen. In the complex organization of modern technological society, there has been a necessary division of thought and labor, and yet the carefully balanced alloy of the philosopher and the practitioner has been the metal for forging the characteristic Industrial Society of today.

Although, as Dr. Albert Einstein has pointed out, "We now realize, with special clarity, how much in error are those theorists who believe that theory comes inductively from experience,"[1] the immutable demands of science are such that new concepts inevitably bear at least the subjective imprint of past experience, and that, inevitably, there must be a commingling of theory and practice.

In sum, the object and method of the search are clear. They are to establish a true Science of Managing based upon a valid, moral, and ethically acceptable philosophy of management by the impartial observation of social components as discrete entities within and related to a total common purpose. The search,

[1] *Out of My Later Years*, Dr. Albert Einstein, Philo. Library, N. Y., 1950, p. 72.

especially in business and little less in government, has been, and will continue to be, conducted in a climate of reality for:

> Life cannot wait until the sciences may have explained the universe scientifically. We cannot put off living until we are ready. The most salient characteristic of life is its coerciveness: it is always urgent 'here and now', without any possible postponement . . .[2]

III

There have been many definitions of "Scientific Management," or as we prefer to phrase it, of a specific "Science of Managing." Mostly, they have been reflections of their time and frequently have served the needs of brevity more than of understanding. Yet awareness of the need for managing skills dates back, of course, beyond the beginning of recorded history. There are three phases, however, which, at the risk of rather obvious over-simplification, can be divided in time.

Before the nineteenth century, industry and business as we know it today were basically craft, individual, or, at best, guild matters. Dominating other aspects and counting above other endeavor usually was the organization and pursuit of territorial conquest. The energy of creative leadership was devoted in large part to the expansion of geographic horizons and the consequent need to discover, to take over, and to govern new territories.

Thus, there is evidence of "management" in military structures as in the conduct of civilian public affairs, but it was a type of management in which thought over and above the satisfaction of immediate requirements was usually of minimum proportions or impact. This era was perhaps the clearest example of management as an art; that is, the application of knowledge without systemization, and ordinarily on a definitely personalized basis.

The second period, on this scale of classification, began in the early part of the nineteenth century, following closely upon the heels of the introduction of the power loom and the steam engine. Here the first clearly definitive movement toward understanding the managerial, and even the broader social, implications of rapid technological progress was made, although nearly always in the light of just adequate and immediate solutions.

The information necessary to establish a true "Science of Managing" was still not at hand. The implications of constructive and successful steps taken in the direction of sound and rational organization were largely overlooked. Controlled experiment, accurate observation, and statistical correlation of human processes sometimes fell far short of more than lip service. Those who had created the industrial revolution, however, were concerned with the changing world they had brought into being, and the writings indicate at least a growing realization of a serious, impending human problem.

In the last half of the nineteenth century, therefore, they talked hopefully of a "science of management." But, because they dealt only with specific fragments of a complex problem, the principles they accepted are seen by the broader view

[2] Ortega y Gasset, from *Mission of the University*, London, Kegan Paul, 1946, p146.

of hindsight to have been overly narrow and superficial; and the science they established was, by today's dimensions at least, only a quasi-science.

A more serious danger inherent in the limited understanding of the nineteenth century mechanists has been pointed out by Professor A. N. Whitehead:

> A factory with its machinery, its community of operatives, its social service to the general population, its dependence upon organizing and designing genius, its potentialities as a source of wealth to the holders of its stock is an organism exhibiting a variety of vivid values. What we want to train is the habit of apprehending such an organism in its completeness. It is very arguable that the science of political economy, as studied in its first period after the death of Adam Smith (1790), did more harm than good. It destroyed many economic fallacies, and taught how to think about the economic revolution then in progress. But, it riveted on men a certain set of abstractions which were disastrous in their influence on modern mentality. It dehumanized industry[3]

Many historians mark the first decade of the present, or twentieth, century as the beginning of the work of investigating the principles of management along lines which provide statistical validity; although even casual analysis, especially of papers on cost accounting and on work analyses before such bodies as the American Society of Mechanical Engineers, shows clearly that the foundation studies for such investigation had been developing for at least twenty years previously.

The examination of work skills in terms of output, which had thus received growing attention in the last quarter of the preceding century, was progressively restudied and carried forward with remarkable diligence and with increasingly startling results.

Resources in men and materials were expended in a great surge of discovery, but, unfortunately, the comparative abundance of the things man needed to make, to build, and to expand was in effect an embarrassment of riches. In the haste to make mightily, men fell too often into ways of leadership by command and by compulsion. The latent power of the people, which Gustave LeBon termed the "psychological law of the mental unity of crowds,"[4] was accordingly marshalled in a pattern of social revolt, directed at such industrial methods because, while they did produce mightily, they tended to distribute inequitably and to make advances with inadequate regard for human developments.

Students of social status, customs, and trends became concerned with the creation, maintenance, and growth of the business, and specifically of the industrial enterprise as perhaps the most significant of current social phenomena. They realized, at last, that the total equation of civilization was heavily, perhaps even dominantly, affected by its business and industrial factors.

It is within the last forty years, in consequence, that we have at last turned our primary, rather than our incidental, attention from *the* man, *the* machine, *the* product, to total enterprise, and even to Industrial Society, as an entity.

Those concerned with the Science of Managing are going back, now, over the

[3] *Science and the Modern World*, A. N. Whitehead, MacMillan Co., 1925.
[4] *The Crowd*, Gustave LeBon, Ernest Benn Ltd., London, 1952, p. 24.

accumulated knowledge and experience of centuries with new attitudes to discover the basic principles and patterns "which, like the great religions of the world, have the power to arouse the faith and hold the support of the great body of individual men and women throughout the ages."

IV

The birth of the organized movement in search of a rational and cohesive science of management is generally credited to Frederick Winslow Taylor.[5] His book, *The Principles of Scientific Management*, published in 1911, and synthesizing and advancing the theses developed in his earlier experiments and writings from around 1880 to that time, seriously upset many traditional concepts of management.

It is interesting to study Taylor's development as, in the course of a long and productive life, both as a practitioner and engineer in and as a writer on the work of Managing, his attention and emphasis shifted from the exploration of discrete facets of industrial processes to a search for underlying principles which governed the operation of those processes. It is, in a sense, the condensed pattern of the development of a real management science.

Taylor was born in 1856 and, after a rather cursory education, became apprenticed to the Enterprise Hydraulic Works in Philadelphia in 1874. He became concerned with the serious gap between the potential output and the actual output of shop workers and when he became a foreman, he determined to study the means of increasing productivity.

His early attempts were penetrating studies of the machine as a unit of productivity. Carefully controlled experiments were made on metal cutting techniques at Bethlehem Steel, at first alone and later with the aid of such associates as Carl Barth, Henry Gantt, and William Sellers. Every conceivable variation in speed, feed, depth of cut, and kind of tool was made and an empirical understanding of optimum combinations was established. And, in parallel with these empirical advances, it is significant that Taylor also sought the aid of able mathematicians of the day to find theoretical explanations and formulas from the abundant data with its complex and baffling variables, which their work amassed.

Encouraged by the success of the metal cutting work, Taylor made many other studies which dealt with the techniques of production. His work, during this period, was characterized primarily by his concern with end results, yet also by significant parallel attempts to deduce basic meanings. However, just as

[5] References in this paper to individual practitioners and writers on Scientific Management are made solely to indicate by example the *progression of ideas* that developed with changing times. This paper is not an attempt to mention systematically even the outstanding pioneers of the "Management Movement." For a systematic listing of such pioneers, and for a comprehensive presentation of their contributions, see the forthcoming Golden Book of Management, prepared by CIOS, the Comite International de l'Organization Scientifique, with headquarters at Geneva, Switzerland, which now represents the organized management societies of some twenty-four free nations at the international level. Any attempt to list even the leading current authorities in the field would simply require another large book in itself.

the metal cutting experiments dealt with what was happening more than why it happened, he failed, for the moment, to perceive the full general implications of his studies as an example of one of the elemental processes of scientific management.

Paralleling Taylor's and other early explorations, Frank and Lillian Gilbreth undertook a remarkable series of studies, which were distinguished by the fact that they definitely recognized and recorded with persuasive clarity that the basic unit of productivity had to include the worker as well as the machine and on a quantitative and specific basis. In 1909, for example, they published a work entitled *A Bricklaying System* and, in 1911, followed it with the more comprehensive *Motion Study*.

Their work, while of tremendous importance in creating an understanding of motion economy, of the techniques of increasing output by reducing incremental effort, and of the tools of measurement, is of deeper and more lasting importance in that it showed the significance of integrated thinking. They found and preached for all to know that it was the optimum combination of worker skills and machine operation, rather than the best of either alone, that could narrow the gap between potential and realized production.

The full import of the contribution of the industrial psychologist was not at first recognized. In reviewing the history of the American management movement, Colonel Lyndall F. Urwick gave this warm evaluation of the Gilbreth team:

> If they (the ASME) had not been (aware of human problems involved)—and Taylor either failed to encounter, or to recognize the significance of, the early work in industrial psychology contributed by Walter Dill Scott, Hugo Munsterberg, and others—there was the amazing fact that one of them, Frank Bunker Gilbreth, happened to fall in love with a girl who was a psychologist by education, a teacher by profession, and a mother by vocation. I know of no occurrence in the whole history of human thought more worthy of the epithet "providential" than that fact. Here were three engineers—Taylor, Gantt, and Gilbreth—struggling to realize the wider implications of their technique, in travail with a 'mental revolution,' their great danger that they might not appreciate the difference between applying scientific thinking to material things and to human beings, and one of them married Lillian Moller, a woman who by training, by instinct, and by experience was deeply aware of human beings, the perfect mental complement in the work to which they had set their hands.[6]

As such work and that of many other pioneers progressed steadily, as is significant that Taylor, too, increasingly appreciated the fuller meanings of his work, and out of such developing awareness he offered this concept of "the manager" in *The Principles of Scientific Management*:

> These *new* duties of the manager are grouped under four heads:
> 1. They develop a science for each element of man's work, which replaces the old rule-of-thumb method.
> 2. They scientifically select and then train, teach, and develop the workman, whereas in the past he chose his own work and trained himself the best he could.

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[6] *Management's Debt to the Engineers*, The ASME Calvin W. Rice Lecture, Colonel Lyndall F. Urwick.

3. They heartily cooperate with the men so as to insure all the work being done in accordance with the principles of the science which have been developed.
4. There is an almost equal division of the work between the management and the workmen. The management takes over all work for which it is better fitted than the workmen, while in the past almost all of the work and the greater part of the responsibility were thrown upon the men.

And, in that same treatise, he ably summarized the powerful concepts which he had evolved and clarified in his more than thirty-five years of resourceful, persistent, and classified studies in these memorable words:

The writer is one of those who believes that more and more will the third party (the whole people), as it becomes acquainted with the true facts, insist that justice shall be done to all three parties (employer, employee, public). It will demand the largest efficiency from both employers and employees. It will no longer tolerate the type of employer who has his eye on dividends alone, who refuses to do his full share of the work and who merely cracks the whip over the heads of his workmen and attempts to drive them into harder work for low pay. No more will it tolerate tyranny on the part of labor which demands one increase after another in pay and shorter hours, while at the same time it becomes less, instead of more, efficient.

And the means which the writer firmly believes will be adopted to bring about, first, efficiency both in employer and employee, and then an equitable division of the profits of their joint efforts, will be scientific management, which has for its sole aim the attainment of justice for all three parties through impartial scientific investigation of all the elements of the problem. For a time both sides will rebel against this advance. The workers will resent any interference with their old rule-of-thumb methods, and the management will resent being asked to take on new duties and burdens; but in the end the people, through enlightened public opinion, will force the new order of things upon both employer and employee.

. . . Scientific management does not necessarily involve any great invention, nor the discovery of new or startling facts. It does, however, involve a certain *combination* of elements which have not existed in the past, namely, old knowledge so collected, analyzed, grouped and classified into laws and rules that it constitutes a science; accompanied by complete change in the mental attitude of the working men as well as of those on the side of management, toward each other, and toward their respective duties and responsibilities. Also a new division of the duties between the two sides and intimate, friendly cooperation to an extent that is impossible under the philosophy of the old management . . .

It is no single element, but rather this whole combination, that constitutes scientific management, which may be summarized as:

Science, not rule-of-thumb
Harmony, not discord
Cooperation, not individualism
Maximum output, in place of restricted output
The development of each man to his greatest efficiency and prosperity

The time is fast going by for the great personal or individual achievement of any one man standing alone and without the help of those around him. And the time is coming when all great things will be done by that type of cooperation in which each man performs the function for which he is best suited, each man preserves his own individuality and is supreme in his particular function, and each man at the same time loses none of his originality and proper personal initiative, and yet is controlled by and must work harmoniously with many other men.

Both Taylor and the Gilbreths had thus restated, and indeed proved, two basic concepts which set the pattern of the industrial revolution over a hundred

years before. Adam Smith in *Wealth of Nations*, 1776, suggested the principle of the division of labor:

> This great increase of the quantity of work which, in consequence of the division of labor, the same number of people are capable of performing, is owing to three different circumstances; first, to the increase of dexterity in every particular workman; second, to the saving of the time which is commonly lost in passing from one species of work to another; and lastly, to the invention of a great number of machines which facilitate and abridge labor, and enable one man to do the work of many.

This, of course, was the historical precedent of the work in the early 1900's which was concerned with the organization of production skills and which has since been extended and enlarged by the contributions of Engstrom, Maynard, Segur, and many others in this country and by fellow scholars and practitioners in these fields abroad.

Charles Babbage, British mathematician and scholar, provided the second governing concept of *transference of skill* in his *Economy of Machinery and Manufacture*, published in 1832:

> That the master manufacturer, by dividing the work to be executed into different processes, each requiring different degrees of skill and force, can purchase exactly that precise quantity of both which is necessary for each process; whereas, if the whole work were executed by one workman, that person must possess sufficient skill to perform the most difficult, and sufficient strength to execute the most laborious, of the operations into which the art is divided.

It is interesting to compare the statements of Babbage and Taylor. Superficially, they seem to say much of the same; actually, Taylor's four principles of managing introduced the means by which Babbage's "transference of skill" *might be accomplished*. While many still think of Taylor as a seeker of cold efficiency, the true scope of his work is more accurately envisioned in this prefatory summation from his 1911 book:

> The principal object of management should be to secure the maximum prosperity for the employer, coupled with the maximum prosperity for each employee.
>
> The words 'maximum prosperity' are used, in their broad sense, to mean not only large dividends for the company or owner, but the development of every branch of the business to its highest state of excellence, so that the prosperity may be permanent.
>
> In the same way maximum prosperity for each employee means not only high wages than are usually received by men of his class, but, of more importance still, it also means the development of each man to his state of maximum efficiency, so that he may be able to do, generally speaking, the highest grade of work for which his natural abilities fit him, and it further means giving him, when possible, this class of work to do . . .
>
> The majority of these men (employers and employees) believe that the fundamental interests of employees and employers are necessarily antagonistic. Scientific management, on the contrary, has for its very foundation the firm conviction that the true interests of the two are one and the same; that prosperity for the employer cannot exist through a long term of years unless it is accompanied by prosperity for the employee, and vice versa; and that it is possible to give the workman what he wants—a low labor cost—for his manufactures.[7]

[7] *The Principles of Scientific Management*, Frederick W. Taylor, 1911, p. 9–10.

Another milestone of considerable stimulative value was passed in connection with the testimony of Harrington Emerson and other engineers before the Interstate Commerce Commission. In October of 1910, Louis D. Brandeis and Henry L. Gantt brought together a group of engineers to choose the most suitable desigation for the new philosophy of management.

Mr. Brandeis was the principal attorney of freight shippers who were fighting the imposition of railroad rate increases. The essence of his strategy was to prove by competent testimony that a method existed whereby the railroads could not only reduce rates but could, at the same time, reduce costs and increase wages. He realized that the case would be strengthened if all his witnesses called the same things by the same names and would agree on a single name to designate the system of management they represented.

Mr. Emerson pointed out, under careful questioning by Mr. Brandeis, that the railroads of America could save at least a million dollars a day by the application of scientific principles to the operation of their business. Sudden realization among business leaders everywhere that the then proudest industrial achievement, the system of railroads, was actually something less than the flawless gem of American enterprise, brought at last the needed widespread attention and support the management movement had lacked and gave the newly chosen name, Scientific Management, an official introduction.

A conference of some three hundred businessmen, consultants, and educators was called in 1912 at Tuck School at Dartmouth to discuss the possible courses of action uncovered by such new avenues of management thinking. The deliberations of these historic sessions, powerfully preserved by Harlow Persons, are considered by many scholars to mark the "Charter" of an *organized* "Management Movement" in this country within which to progress and cumulate individual contributions in a meaningful way.

Almost overnight, "Scientific Management" became a matter of public concern and open debate. As so often happens, however, enthusiasm outran understanding. Although resistance to excesses was prompt, the "efficiency expert" became the apostle of exploitation in the eyes of the great body of labor, rather than a leader for mutual progress and agreement.

Among management people themselves there was still a further dissimilarity of enthusiasm, reflecting that human nature is the common characteristic of both the managerial and the individual worker. Traditionalists regarded the work of the management investigators as so much pap and set out on an active de-bunking movement, rooted in an attempt to maintain the only ways of work they could or wanted to understand. Progressives refused the challenge of a pointless battle and began to re-evaluate their responsibilities as managers.

Contrast these two answers to a survey conducted by the American Society of Mechanical Engineers and quoted in the first of that Association's *Ten Year Progress in Management Reports*, published in 1912, which from 1912 to 1932 were under the distinguished guidance of L. P. Alford. In response to a request for a definition of the new element in the art of management, the traditionalist viewpoint held:

I am not aware that a *new* element in the art of management has been discovered . . .

There have been no new discoveries in scientific management of industrial institutions. Common-sense men have used common-sense methods always. The term 'scientific management' is a catch-word which assumes that industrial institutions have not been scientifically managed—which is not the case. My experience and the experience of my friends has been that there has been no new element injected into the art of management.

In the writer's opinion there is very little that is new about it (the art of management). There is hardly any part of it that has not been practiced by managers for the past 100 years. The trouble is there are not enough managers with sufficient initiative to set the system moving properly.

. . .the problem presented is not the adoption of something entirely new; but rather the extension to every detail of our work of something which we have already tried.

This was the classic pattern of the resistance. The writer made categorical admission of two basic hazards in the path of industrial development—the lack of adequately trained and inspired managers and the need for the extension of scientific method to the over-all enterprise. He offered neither solution nor alternative and apparently was willing to believe that the changing nature of management was a fictitious academic dream.

The ASME Committee, with J. M. Dodge as Chairman and Alford, himself, as Secretary, rejected this concept of "impossibility" and selected from among the many favorable responses one which seemed best to convey the nature of the then so-new "science":

The best designation of the new element I believe to be 'scientific management.' This term already has been adopted quite generally and although frequently misused, carries with it the fundamental idea that the management of labor is a process requiring thorough analytical treatment and involving scientific as opposed to 'rule-of-thumb' methods.

The writer ventures to define the new element briefly, but broadly, as: The critical observation, accurate description, analysis, and classification of all industrial and business phenomena of a recurring nature, including all forms of cooperative human effort and the systematic application of the resulting records to secure the most economical and efficient production and regulation of future phenomena.

Stripped of technicalities the method of the modern efficiency engineer is simply this: First, to analyze and study each piece of work before it is performed; second, to decide how it can be done with a minimum of wasted motion and energy; third, to instruct the workman so that he may do the work in the manner selected as most efficient.

The Taylor System is not a method of pay, specific ruling of account books, not the use of high-speed steel. It is simply an honest, intelligent effort to arrive at the absolute control in every department; to let tabulated and unimpeachable fact take the place of individual opinion; to develop 'team play' to its highest possibility.

As we conceive it, scientific management consists in the conscious application of the laws inherent in the practice of successful managers and in the laws of science in general. It has been called management engineering, which seems more fully to cover its general scope than a science.

The 1912 (ASME) Progress Report continues in reference to this second letter:

These quotations convey the ideas of a conscious effort to ascertain and study facts and systematically to apply them in instructing the workmen and in controlling every department of industry. Setting these against the underlying principle of the transference of skill, we conceive the prominent element in present-day industrial management to be: *The mental attitude that consciously applies the transference of skill to all the activities of industry.*

The work of the committee, advanced and comprehensive though it then was, of course still fell short of a full appreciation of the basic nature of scientific management. They rejected as inaccurate and muddled a suggested approach to a specific means of putting into practice the "attitude that consciously applies the transference of skill." In the light of present theory and practice, however, this statement by an unnamed correspondent of the Committee is neither fatally inaccurate nor particularly muddled in conceptual understanding, even though the "functional foreman" concept which was advocated to permit specialization in skills, has since been found to be less desirable than the single foreman backed and aided by functional staff specialists:

> The regulative principles of management along scientific lines include four important elements:
>
> a. Planning of the processes and operations in detail by a special department organized for this purpose.
> b. Functional organization by which each man superintending the workman is responsible for a single line of effort. This is distinctly opposed to the older type of military organization, where every man in the management is given a combination of executive, legislative, and judicial functions.
> c. Training the worker so as to require him to do each job in what has been found to be the best method of operation.
> d. Equable payment of the workers based on quantity and quality of output of each individual. This involves scientific analysis of each operation to determine the proper time that should be required for its accomplishment and also high payment for the worker who obtains the object sought.

As a result of the interest in the railroad rate cases, of the Dartmouth meeting, and of the generally increased attention of engineers and the public, 1912 and 1913 saw the formation of many new associations. Most of them, at the time, were essentially either splinter groups broken off from the basic ASME body or else newly organized as a result of somewhat different objectives or the desire of specialists to emphasize special facets of the movement.

One of the most important of these new societies was also one of the most short lived. It was important because it numbered among its members of record that kind of mixture of outstanding industrial executives and business managers, as well as management scholars, theorists, educators, economists, and publicists, which has allowed theory and practice to crossfertilize each other as the American Industrial Society has evolved. It's name, the Efficiency Society, was unfortunate since the word "efficiency" had begun to have a rather caustic effect on the public. It's reasons for failure were, however, somewhat more fundamental. Charles Buxton Going, Managing Editor of *Engineer Magazine*, in outlining the purposes and objectives of the Society wrote:

> The essence of the Efficiency Movement is insistence upon a determination of standards of achievement—equitable and reasonable standard by which the ratio of useful result secured to the effort expended, or the expense incurred in any given case, may be compared with the ratio that should exist in a normal utilization of the agencies at hand. Efficiency does not demand nor even encourage strenuousness. It does not impose nor even countenance parsimony. It merely demands equivalence, equivalence between power supplied and

work performed; equivalence between natural resources utilized and products obtained; equivalence between vital opportunity and individual or national health; equivalence between attainable degrees of security and the actual proportion of casualties; equivalence between production capacity and finished product.[8]

One can hardly take issue with the "insistence" or the "demands." Certainly they are only objectives which are approachable and beneficial to society.

Yet, there is an air of coldness and of compulsion about this statement which could hardly be expected to win understanding or reduce antagonism. What seemed to be essentially lacking was an adequate awareness that the man at the machine might value and protect his own conception of his own dignity—that in the last analysis any hope for a more efficient world would necessarily have to depend on making the worker aware and voluntarily appreciative of the fact that although his objectives and those of the enterprise might normally be different, both sets of objectives could only be achieved together; that is, that their desires were not mutually exclusive, merely different.

Requoting Taylor on this point: "Scientific Management, on the contrary, has for its very foundation the firm conviction that the true interests of the two are one and the same . . ."[9] Although, of course, Taylor generalized—the "true interest" often being overshadowed by the *apparent* interest—he clearly appreciated the nature and magnitude of the human problem.

So, also, did others. A. Hamilton Church and L. P. Alford, for example, wrote:

Some of the conditions of personal effectiveness are these: The individual must feel leadership; have adequate encouragement and reward; be physically fit and under good physical conditions; and receive a definite allotment of responsibility.

These conditions apply not only to the operative force but to all grades of employees. In fact, some of them apply with greater urgency to the man 'higher-up' than to the actual worker.

The truth is, of course, that no single element of a system, or even a combination of half a dozen of such elements . . .more than touch the fringe of the questions. Highly organized systems may coexist with fine esprit de corps but the latter is not dependent on any form of system or organization.

Of all the conditions controlling a fine working atmosphere, leadership probably plays the most important part . . .The weakness of one prominent school of management doctrine is that it pretends that it has superseded leadership by substituting therefor elaborate mechanism. Such a contention betrays a complete misapprehension of how men are constituted and of what the true functions of elaborate mechanisms really are. All such mechanism is but a collection of mechanical tentacles or feelers to enable the controlling mind and spirit of the management to be in several places at once. If personality behind these tentacles is a feeble one, the mechanism will not supplement its deficiencies in the slightest degree.[10]

This was the visionary concept which, in the hands of those who were to carry on the work, has been embellished and amplified as one of the basic tenets of the "Science of Managing."

[8] *The Efficiency Movement. An Outline*, Charles Buxton Going, Efficiency Society, Inc., Transactions, 1912, Vol. I, p. 13.

[9] *The Principles of Scientific Management*, Frederick W. Taylor, 1911, p. 9.

[10] *The Principles of Management*, A. Hamilton Church & L. P. Alford, American Machinist for May 30, 1912.

It found expression in the formation of such associations as The Society to Promote the Science of Management and the National Association of Corporation Schools, the later group devoted primarily to the problems of Training in industry. Somewhat later, in 1917, the growing interest in connection with war work led to the formation of the Society of Industrial Engineers.

Soundly conceived, these organizations grew in prestige over the years. The first and third eventually became the present Society for the Advancement of Management, and the second, also after mergers with others, became the present American Management Association.

In reviewing the rise and fall of various management organizations, a possible key to their success or failure may be found in the answers to Professor Dwight Waldo's questions:

> Are students of administration trying to solve the problem of human cooperation on too low a plane? Have they, by the double process of regarding more and more formal data over a wider and wider field of human organization, lost insight, penetration? Is formal analysis of organizations without regard to the purposes that inspire them but a tedious elaboration of the insignificant?[11]

Where the sights have been properly set, and the objectives honestly derived from the inherent obligations imposed on the work by the needs of society in general, and of individual human beings in particular, management associations have flourished. They are accepted now as a necessary and desirable professional component of our technological Industrial Society and are progressively expanding their contributions based on their sound foundation.

V

The years between 1912 and 1922, the date of Alford's second ASME Progress Report, were years of international unrest and of world-wide War. Demands on current material and human resources had required almost the total attention of management thinking, and the theoretical aspects of the report were, therefore, very nearly restatements of the 1912 Report. *Practically*, however, it was an era of great advances, since the unprecedented demands of the war effort required the *application* of every organizational and functional skill at hand.

One of the characteristics of a science, the alternate play and shifting dominance of theory and practice as demanded by necessity, then became evident in the Science of Managing.

If it is possible to assess against one man the stimulation for making the theoretics of management into working realities during World War I, such appraisal would undoubtedly point to Bernard Baruch. Both in the specific structure of his War Industries Board, as it was finally constituted in 1918, and in substituting centralized and authoritative governmental planning for the free market and the law of supply and demand which in effect enforced efficient managerial attitudes by such devices as rigid priorities, fixed prices, and absolute

[11] *The Administrative State*, Dwight Waldo, The Ronald Press Co., p. 211.

schedules, he gave industry little choice but to streamline and clean house or to fail. Heavy production schedules and severely limited profits thus practically forced these industry leaders, who had not done so through foresight and conviction, to turn to scientific management as a means of survival.

Baruch seemed peculiarly gifted in his ability to look at the mobilization effort as essentially an economic proposition. He demanded, and finally received, authority to make decisions and enforce them over the total field of supply and demand of not only the materials of war itself, but over the total economy. Most importantly, however, he substantially avoided the inherent dictatorial dangers of such a concentration of power by delegating his authority to subordinates in order to put decision-making in the hands of experienced economic and industrial experts who were close to the scene of action.

Out of the chaos of the 1914–1917 period, a time when the President's Advisory Commission was in the untenable position of being asked to make decisions but prevented by charter from enforcing the decisions, the United States thus finally achieved the integration of its aims and its capabilities, aided, of course, by that terrible but effective commonness of purpose and spirit which the fires of War so rapidly forged. How it was done is summed up by James Tyson in these words:

> The great principle followed throughout the Board's dealing with industry was that of voluntary cooperation with the big stick in the closet. The biggest problem was to increase production so as to raise the output of industry up somewhere nearer the tremendous demands of the government. For this reason it was necessary to give business every encouragement, by allowing a margin of profit and also by attempting to arrive at an agreement with each trade before imposing conservation or other regulations . . .From the time of his early attempts to bring producers together . . .he (Baruch) followed this policy of close alliance rather than one of arbitrary control.
>
> Perhaps no better appraisal of the final forms of this cooperating could be found than the observation of Paul Von Hindenburg in his memoirs, when he said of American Industry: 'Her brilliant, if pitiless, war industry had entered the service of patriotism and had not failed it. Under the compulsion of military necessity a ruthless autocracy was at work and rightly, even in this land at the portals of which the Statue of Liberty flashes its blinding light across the sea. They understood war!'[11]

The principles of leadership include of necessity an understanding of the limitations of those who are led.

Whether these limitations are the result of tradition, of prejudice, or of apathy, they exist and have to be dealt with in an atmosphere of reality. That not every man wants to be or is capable of being captain of his own ship was demonstrated in the attempts of Edward and Lincoln Filene, who were both early theorists and early practitioners of scientific management, to put their famous Boston department store on a cooperative basis.

Edward Filene, described by one of his associates as an "ingenuous and ingenious idealist" embarked on a program in 1912 which, in the cold light of hindsight, was as noted for its impracticality as it was for its humanity. He and

[11] *The War Industries Board, 1917–18*, James L. Tyson, Supplement to Fortune for September, 1940, p. 16.

his brother tried desperately to encourage the interest and active participation of the workers in the enterprise by establishing a Cooperative Association together with plans for the eventual transfer of all stock to employees. Indifference toward the exercise of power and resistance toward assuming responsibility for their own corporate destiny on the part of the workers was so startlingly apparent that Lincoln Steffens facetiously suggested that Filene might have to hire some agitators to put his program across.

The effort to put the enterprise in the hands of the workers continued unsuccessfully for more than ten years. It failed, not because of basic violations of ethical standards or of public and employee interests, but because of the failure of the Filenes to establish a system of communication with their employees which would allow them to determine the employees' concept of the manager's and worker's common interests.

The failure, a personal disaster for Edward Filene, made nevertheless two significant contributions to the business community. It showed, that managing is a mantle of responsibility not willingly accepted by all people. Moreover, it demonstrated the fundamental need for thorough and undistorted study of all the facets of a problem. Thus, it highlighted, for management theorists and practitioners, the importances of these three elements of business managerial thinking: Thinking ahead, thinking through, and thinking whole.

VI

The awakening of the true nature of the modern American leadership process came about in the period following World War I. Now, at last, acute realization of the closely geared relationship of the business enterprise as an integral part of, and at the same time as a significant contributor to a general pattern of social development, was broadly achieved.

The story of this awakening in industry is, in large part, vividly reported in the gifted observations and writings of Mary Parker Follett. Her papers and lectures covering some thirty years of uniquely contributive observation are remarkable both for their breadth of application and for the penetrating understanding of motive and need.

She realized that the true quality of modern business Leadership stems from the appreciation of the basic needs and aspirations and of the mutual dependence of men in a complex social organism. Throughout her long career Miss Follett was fortunate in having the friendship and advice of the many industry and business leaders, including especially many in the management of the New England Telephone and Telegraph Company, who shared with her the benefit of their experience, and for whom she was able to express the meaning and import of their work. Her relationships with the telephone system were especially fortunate because the gifted early managerial work of Theodore Vail, president of the parent company in the Bell System, had provided an environment of management by well-defined and far-seeing policies within a clearly-designed functional organization structure that has in essence survived to this day and that was peculiarly appropriate for her perceptive observation and advice.

Mary Follett received her formal education at Radcliffe College. Her work there and her subsequent contributions earned her a place among the College's fifty most distinguished graduates. Metcalf and Urwick, in their collection of her papers, *Dynamic Administration*, attempted to define the special quality of her attitudes which gave her work such significance. It is repeated here because it is, in a way, the definition of a rare managerial trait:

> Mary Follett's outstanding characteristic was a facility for winning the confidence and esteem of those with whom she came in contact; she established a deeply-rooted understanding and friendship with a wide circle of eminent men and women on both sides of the Atlantic. The root of this social gift was her vivid interest in life. Every individual's experience, his relations with others and with the social groups—large or small—of which he was a part, were the food for her thought. She listened with alert and kindly attention; she discussed problems in a temper which drew the best out of the individual with whom she was talking. The strength of the personal associations she thus built up were remarkable.

Miss Follett progressed from community activities to social work, and from there to vocational guidance and finally to business and industrial organization. In the latter work she drew heavily upon her experience in practical psychology, and her lectures and papers were strongly woven with the threads of understanding and sympathy. Her philosophy was, of course, the synthesis of the studies of many people concerned with the theory of organization.

In the meantime, Mrs. Gilbreth, like many others prominent in the history of the management movement, continued her analysis of industrial problems, delineating specific worker and manager attributes which contributed to the balance of the business economy. At the same time comprehensive full-length books began to supplement the shorter conference-type papers as the building blocks of the literature of Management. Thus, such volumes as Mooney and Reiley's *Onward Industry*, and later their *Principles of Organization*, Fayol's *Industrial and General Administration*, Barnard's *The Functions of the Executive*, and Brown's *Industrial Organization* typify the magnitude and scope of the source material to which Miss Follett added her own observation and imagination. It is noteworthy also that these writers were practicing industrialists, two from General Motors, one from the mining industry, one from the Bell Telephone System, and the last from the Johns-Manville Corporation.

Such writings and those of many other leading industrialists, consultants, and educators afforded a firm base for the present comprehensive literature in this field. Thus, the following brief quotations from the Follett papers are, in fact, representative of the creative outpouring of an era rather than of a single person:

On conflict:

> As conflict—difference—is here in the world, as we cannot avoid it, we should, I think, use it. Instead of condemning it, we should set it to work for us. . . There are three main ways of dealing with conflict: domination, compromise, and integration. Domination, obviously, is a victory of one side over the other. This is the easiest way of dealing with conflict, the easiest for the moment but not usually successful in the long run.
>
> The second way of dealing with conflict, that of compromise, we understand well, for it is the way we settle most of our controversies. . . Yet no one really wants to compromise, because that means a giving up of something. Is there then any other method of ending

conflict? There is a way beginning now to be recognized at least and occasionally followed: when two desires are *integrated*, that means that a solution has been found in which both desires have found a place, that neither side has to sacrifice anything.

On Business as an integrative unity:

It seems to me that the first test of business administration, of industrial organization, should be whether you have a business with all its parts so coordinated, so moving together in their closely knit and adjusted activities, so linking, interlocking, interrelating, that they make a work unit—that is, not a congeries of separate pieces, but what I have called a functionwhole or integrative unity.

On the nature of Power:

So far as my observation has gone, it seems to me that whereas power usually means power-over, the power of some person or group over some other person or group, it is possible to develop the conception of power-with, a jointly developed power, a co-active, not a coercive power.

On the Psychology of consent and participation:

Many people are now getting beyond the consent-of-the-governed stage in their thinking, yet there are political scientists who are still advocating it. And, indeed, it is much better to have the consent of the governed than not to have it... but we are also recognizing today that it is only a first step; that not consent but participation is the right basis for all social relations.

The literature of management during the period between the two World Wars shows how completely Miss Follett's views gave expression to the framework which had become a fundamental part of the thinking of industry.

Although it would be impossible in this limited treatise to give even an indication of the wealth of writing done during this period, no paper on the history of management concepts would be possible without at least a partial mention of the many contributions made here and abroad. It is interesting to note the significant degree to which many of these fundamental books were the work of men whose professional careers had been directly concerned with business operations. Few of them were "writers," and it is almost by incidental and fortunate circumstances that their work is so readable, not discounting, of course, that clarity of conviction and of purpose are themselves no mean aids to such clarity of presentation.

Dr. Harry Arthur Hopf, whose own work will be discussed at a later point, listed twelve indispensable books, chiefly of this general period, and his reviews on five are so indicative of the form which the evolving "science of managing" was assuming that they deserve to be quoted here in part:[18]

The Philosophy of Management, by Oliver Sheldon (of Great Britain), was "written from a broad perspective; it stresses the importance of scientific and ethical principles, gives an

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[18] *Soundings in the Literature of Management*, Harry A. Hopf, Hopf Inst. of Management, Publication No. 6.

excellent exposition of the social and industrial background, and deals in an authoritative manner with fundamentals of management."

Industrial and General Administration by Henri Fayol (a leading French manager of mining and industrial firms). "His masterly analysis of the essential functions of a business enterprise, his selection among them of administration for special treatment leading to a statement of five underlying principles, and his advocacy of the latter in the form of the Administrative Doctrine, combined to lay the foundation for a new school of thought known as 'Fayolism.' "

Top-Management Organization and Control, by Holden, Fish, and Smith (with combined experience in educational and industrial circles in California), "deals with a field which has hitherto been little explored. . . On the strength of their research study of the management policies and practices of thirty-one leading American industrial corporations, the authors have performed the signally valuable service of bringing together, in admirably organized form, a great amount of factual and interpretive material bearing upon some of the most important and complex management problems with which large-scale industrial organization is confronted."

The Principles of Organization by Mooney and Reilly (of the General Motors organization) is a scholarly work dominated to a large extent by the historians' approach. It covers the history of the management effort as it has applied to the organization of the state, the church, the army, and industry. "This is not a work which may be readily mastered. Its careful study will, however, supply the reader with a sound framework of principles which will serve excellently the purpose of orientation."

Lectures on Organization by Russell Robb is a collection of the lectures delivered in the course on industrial organization at Harvard University. "The author, a distinguished engineer (connected with the Stone & Webster engineering, financing, and management organization) who died in 1927, brought admirably to expression in these lectures a varied experience distilled into a philosophy which, taken as a whole, constitutes perhaps the single most authoritative and appealing exposition stemming from an American to be found in the literature of organization.

Other indispensable books listed by Dr. Hopf were *The Design of Manufacturing Enterprises* by Walter Rautenstrauch, *Industrial Organization and Management* by Ralph Davis, *Industrial Management* by Lansburgh and Spriegel, *Budgetary Control* by James McKinsey, *Personnel Management* by Scott, Clothier, Mathewson, and Spriegel, *Functions of the Executive* by Chester Barnard, and *The Art of Leadership* by Ordway Tead. The role of these authors is also interesting, showing how such universities as Columbia, Ohio State, and Northwestern, as well as industrial firms, were centers of thought of the steadily evolving "science of managing."

In reading these books, the difference in emphasis in fundamental thinking during the twenties and thirties compared to that of the first two decades of the century is apparent. The essence of a deeper *philosophy* of Scientific Management was gradually being distilled and assembled out of the diverse objectives which had been the goals of early investigators. Over-all planning and measurement were replacing the patchwork approach, and though detailed studies of particular situations were necessarily continued, they were increasingly referenced to the framework of the total social scene.

The basic developments of the period were those of bringing into closer blend a proper mixture of the workers' and managers' individual emotional needs and the requirements of an industrial enterprise constituted for the rigors of com-

petitive life in an increasingly complex technological environment. The Science of Managing thus began to appreciate and encompass the techniques of multiplying human skills as well as mechanical power.

VII

When the final review of the history of management during this half-century is written, perhaps it may seek a representative of the movement who, in his work and in his writing, symbolizes the search for the basic and irrefutable principles of the science. They will need to look no further than Dr. Harry Arthur Hopf. Except by an extensive first-hand study of his writings, which unfortunately consist of many separate talks and articles rather than bound volumes, it is impossible to gauge even closely the remarkable gifts he left as a legacy to the student and practitioner of today.

Dr. Hopf, of English birth, came to America in 1898. His first job as a foreign language stenographer for an insurance company was an education in the unending frustration that was the normally accepted part of the non-management employee in business and industry of the time. He observed the disorganization of enterprises devoted solely to the demands of day-to-day problems, the dissatisfactions that came from indecisive or arbitrarily decisive management, the absence of rewards and compensation related even vaguely to effort and contribution at every level.

Writing in *Net Results*, a regular publication of the Institute of Management he later established, Dr. Hopf said:

> With courage (or was it foolhardiness?) and vigor I attacked several situations literally crying for improvement. It was then that I learned for the first time that the way of the reformer is hard, for it required years of the most arduous effort to win a sympathetic hearing for any suggestions. And with the advent of the new life insurance laws in New York State in 1907, my company, in common with other similar institutions, apparently surrendered itself to a case of paralysis of management which was destined to persist for some years . . .

Arduous effort was the normal way of life for Dr. Hopf. His activity as an industrial and business advisor, his participation as founder and pilot of expanded management society activities, his work as a government consultant during both major wars, his seemingly endless capacity to study, to understand, and to offer solutions for the basic problems of the management science paint a picture of nearly legendary proportions.

Out of his many and varied contributions, two, perhaps more than any others, have earned him an enduring place in the annals of the management movement. Dr. Hopf, in his studies of the life insurance business was concerned, of course, with the problem of net efficiency. He noted that the criteria of success were universally related to size, and that these criteria were both wrong and potentially disastrous. Investigations into other industries revealed that this universality was not confined to the insurance business, but that nearly everyone engaged in industry just assumed that the bigger they were, the better, the more efficient, and the more secure they were.

At the Sixth International Congress for Scientific Management, held in London in 1935, he suggested that the time was ripe for the strengthening of the science of management and its transformation to the more inclusive one: optimology—the science of the optimum. In this talk Dr. Hopf said:

> Among the most profound problems with which society must concern itself under present-day conditions is that relating to the determination, achievement, and maintenance of optimal conditions in all types of organized human enterprise. The overwhelming economic disaster, from the effects of which the world is still suffering, halted with ruthless force an era of unparalleled expansion which, in the United States of America at least, assumed proportions indicative of a belief in the feasibility of unlimited growth and unchecked size.
>
> As we falteringly proceed upon the road to recovery, we are faced with new political, social, and economic trends and doctrines which are evidently destined to bring into being forms of organization and control without precedent in our experience, and to call for qualities of cooperation and joint action on the part of businessmen, engineers, social scientists, Government officials, Labour representatives, and others, far beyond any need of the past. Having then, narrowly escaped complete destruction upon the rock of Scylla, are we now being drawn with increasing force into the whirlpool of Charybdis?

Dr. Hopf defined the optimum—for government as well as business—as that state of development of an enterprise which, when reached and maintained, tends to perpetuate an equilibrium among the factors of size, cost, and human capacity which would provide ideal realization of the organizational objectives, and he pointed out that the optimum size was at this state of equilibrium rather than connected, in any way, with bigness alone.

Although Hopf placed no arbitrary limitations on size, he demonstrated that perpetuation of a growing enterprise depended upon the concomitant upward shifting of the equilibrium point. He emphasized that the natural economic barriers to growth are rarely reached, but that the limiting barriers were, most generally, organizational in nature.

In concluding his paper, Dr. Hopf restated a conception of a method of formulating a technique to determine optimal size and relationships which he had originally presented at a meeting of the Taylor Society in 1930. They have become almost a working code of the mid-century science of management:

1. Establish the objectives of the business in comprehensive terms;
2. Define those general policies which should be followed regardless of operating conditions or results;
3. Define the task of management in human terms;
4. Staff the executive group with members who are competent to perform successfully the tasks assigned to them;
5. Furnish the executive group with standards of accomplishment by which performance can be accurately measured;
6. Study operating results and establish trends of accomplishment;
7. Adjust the rate of replacement of members of the executive group in line with requirements for maintaining the standards set;
8. Consider particularly the factor of age in its relation to productive capacity of executives;
9. Analyze all dynamic elements so as to discern the possible operation of the law of diminishing returns with respect to any element, substituting measurement for judgment, wherever possible;

10. Establish the optimal size of organization at the level at which the most favorable operating results can be secured, within the limits of the predetermined objectives and policies and without causing an executive overload at any point in the organization.

Dr. Hopf argued vigorously throughout his life for new perspectives in management. He believed that managing, as such, had to become a special and professional activity. In speaking before the Society of Industrial Engineers in 1933, he quoted Dennison and Willitts'[14] classic definition of a profession:

1. A profession is an occupation which requires intellectual training as contrasted with mechanical skill;
2. A profession employs the fruits of science, uses the scientific method, and maintains an experimental attitude toward information;
3. The professed knowledge is used by its applicant to the service of others, usually in a manner governed by a code of ethics;
4. The amount of financial return is not the chief measure of success;
5. The professions are given public and often legal recognition.

Dr. Hopf examined the work of managing in the light of these criteria, seeking to establish a system of methods which would allow the professional manager to apply the first three as operational techniques. He suggested four basic divisions of the work as forming the dynamics of management: planning, organizing, coordinating, and controlling, a significant and penetrating pioneer attempt to divide the over-all work of the professional manager into primary constituent elements.

In describing the meaning of these four key words, Dr. Hopf continued:

The first of these (planning) involves subdivision of activities to a point where they are within the compass of performance by persons of moderate ability. Failure to observe the requirement is bound to result in the creation of undue supervisory burdens and in obstructions to the smooth flow of operating routines.

The second requirement (organizing) calls for proper relative evaluation of operating units and their grouping along related lines. When the maintenance of arbitarary lines of demarcation among departments, bureaus, divisions, sections, branches, units, etc, comes to be regarded as of greater importance than preservation of the integrity of operating procedures, only disorganization and consequent lack of coordination ensues . . .

The third requirement (coordinating) is the establishment of clear lines of authority, responsibility, and reporting relationships . . . Maintenance of clear lines of authority, responsibility, and reporting relationships under this type of control hinges to a large extent upon integration of the often divergent concepts held by the administrators of the manner in which their relationships to one another shall be composed and of the character of personal supervision over the organization which each shall exercise.

The fourth requirement (controlling) goes right to the root of the problem of administrative coordination. It involves the separation of planning from performance, a sine qua non of effective organization. Progress in undertaking such separation is often attended by conflicts between the points of view of the staff, which is responsible for planning, and the line, which is charged with the accomplishment of satisfactory operating results. Unless

[14] Henry S. Dennison, of the Dennison Manufacturing Company, and Joseph H. Willitts of the University of Pennsylvania.

conflicts can be resolved in favor of cooperative action, sound conditions of administrative coordination are impossible.

Like other great concepts in the philosophies, this one is deceptive in its simplicity. Its value as a working thesis, however, is unquestioned, for, with relatively evolutionary changes, it points yet to most of the basic keys to successful and professional managing.

The range of Dr. Hopf's work encompassed practically all areas of business thinking in both its current and its historical and in both its national and its international aspects. He was at once the theoretician and the practical man, the dreamer and the doer, the pragmatic statistician and the adventurous explorer. These words of Woodrow Wilson on the nature of leadership, written many years before Dr. Hopf's death, well describe this "first universal man of management":[15]

> That the leader of men must have such sympathetic insight as shall enable him to know quite unerringly the motives which move other men in the mass is of course self-evident; but this insight which he must have is not the Shakespearean insight. It need not pierce the particular secrets of individual men: it need only know what it is that lies waiting to be stirred in the minds and purposes of groups and masses of men.[16]

VIII

The modern "Science of Managing" in its evolving full dimensions has come from this historical background, but it is none the less a true product of our own current day.

It is the application of ethical principles by qualified professional managers to the problems of creating and maintaining a complex pattern of order which serves, in optimum fashion, the common interests of the people—that is, of the customers and the public as well as the owners, the executives, and the employees —and of the enterprise itself as a key and characteristic element of today's Industrial Society.

Inherent in the acceptance of such a statement is the responsibility to determine the ethical principles which govern a situation, the qualifications of a professional manager, the pattern of order, and the common interests.

The ethical principles can be briefed simply. To have meaning and to generate faith, a genuine "Science of Managing" can only exist in a climate of liberty, of reason, of morality, and of religion. Outside it—in an air of compulsion, force, materialism, and atheism—it is reduced to impotence and cannot exist any more than the civilization of which it is a part can continue under such circumstances.

[15] As he was called, in memoriam, in Net Results for October, 1949. Alvin E. Dodd, Executive Vice-Chairman, U.S. Council of the International Chamber of Commerce and President Emeritus, American Management Association, and many other business and industrial leaders paid a final tribute to Dr. Hopf through the medium of this little magazine which he and Mrs. Hopf had written for so many years.

[16] "Leaders of Men," Woodrow Wilson, Princeton University Press, Princeton, N. J., 1952.

Deeply grounded in the understanding of our position is the acceptance of the natural rights of the individual as a natural person and that these, coming to him as a person in his own right, transcend in importance these other rights of society as an organized grouping of such individuals where functions come from them to be exercised for them. The individual creates his society not by self-abrogation of these rights, but by his voluntary modification of his liberties derived from them. This is the common interest. This makes possible those kinds of common purpose which justify joint teamwork in organized activities.

Dr. Einstein describes the temperament of the individual as an essentially independent being who willingly becomes a part of and accepts the obligations of a social environment:

> Man is, at one and the same time, a solitary being and a social being. As a solitary being, he attempts to protect his own existence and that of those who are closest to him, to satisfy his personal desires, and to develop his innate abilities. As a social being, he seeks to gain the recognition and affection of his fellow human beings, to share in their pleasures, to comfort them in their sorrows, and to improve their conditions of life . . . The abstract concept 'society' means to the individual human being the sum total of his direct and indirect relations to his contemporaries and to all people of earlier generations. The individual is able to think, feel, strive, and work by himself; but he depends so much upon society—in his physical, intellectual, and emotional existence—that it is impossible to think of him, or to understand him, outside the framework of society.[17]

In order to join his personal aspirations understandingly and satisfyingly with those of the Society of which he is a part, it has become more and more necessary for man to seek for a rational pattern within which to guide and govern his actions. The patterns of order which constitute the goals of this elemental search of all people are achieved by the integration of the results of observation into generalized laws which can be applied with measurably successful results.

Writing of *Management and the American Future*,[18] Lawrence H. Appley, now President of the American Management Association, characterized the professional manager as "... an individual who, because of his training, experience, and competence, is employed to develop and expand the assets and realizations of owners." His horizons might well have been widened to include every conceivable area of human effort where Leadership is a necessity.

The concept of professional managing as Leadership by persuasion rather than by command, and the codification of the Professional Manager's distinct and unique work into the four sub-functional elements of planning, organizing, in-

[17] *Out of My Later Years*, Dr. Albert Einstein, Philosophical Library, New York, 1950, p126. It is significant that in still later writings, Dr. Einstein—now clearly established as one of the foremost scientists of the ages—goes on to reject the concept of a purely random and patternless universe, in the striking phrase, "I cannot believe that God plays dice with the cosmos."

[18] *Management at Mid-Century*, General Management Series No. 169, American Management Association, New York, 1954, p. 5.

tegrating, and measuring, is another of the hard-won milestones in the development of a true Science of Managing.[19]

Speaking of the emphasis placed on developing professional managers by General Electric Company, President Ralph J. Cordiner has said:

> In such an approach is plainly found one deep source of our basic business climate which has made possible the productivity and the better living standards for more and more people which have literally thrust our country into its present position of world-wide leadership and responsibility. But this in turn brings new need to seek how to do a better and more professional job of management.

J. Wilson Newman, President of Dun and Bradstreet, added further understanding of the significance of the Professional Manager concept when he said:

> Free enterprise brings up the subject of free will and decisions. In our country a man can risk his money, time, and skill in business without restraint. That is the way it should be, but as suppliers we are morally obligated to help him with all the friendly guidance we can offer. There is increasing evidence, too, that the new generation of entrepreneurs are better equipped in experience and understanding than were their fathers or grandfathers, although the hazards they face today are certainly greater and more complex. . .
>
> Yet the basic problem is human and emotional rather than statistical. All the fixed operating data can be offset by the intangibles of human nature. The impulse to business risk isn't generated in statistics. It finds life in the eye of the individual who sees an opportunity and measures the risk to achieve it. The quality of his judgment is tested by his ability to overcome obstacles.[20]

IX

During the years of World War II, a new understanding of the problems born of organizations came into being. As in the time of the first World War, the demands of survival speeded the acceptance of the theories of the Science of Managing. Again, practice caught up with theory, but this time a new technology brought time for new horizons.

Again, spurred by the fires of War, the managers of American business, together with the scientists as their partners, and with the spirit of both workers and soldiers to turn ideas and plans to reality, rose to new heights. Global logistics set the demands; the competitive enterprise system, modified once more by governmental Production direction, met the challenge.

[19] The modification of Dr. Hopf's earlier concept of "planning, organizing, coordinating, and controlling," and the rejection of such incomplete classifications as "planning, organizing, and commanding," is apparent. Coordinating is the bringing together of actions; integrating implies unifying to form a complete whole. Control, Dr. Hopf's word, involves the exercise of guiding or restraining power, but such action inherently denies the substance of other sub-functions. Measuring is the process of reviewing performance against predetermined standards for the predicted results of planning, of making such measurements known, and thus of providing a corrective feed-back to the process, so that re-planning, re-organizing, and re-integrating may proceed; thus recognizing the need in organized efforts both for objectives, policies and clear structure on the one hand and for dynamic and vital progress on the other hand.

[20] *Management at Mid-Century*, General Management Series No. 169, American Management Association, New York, 1954, p. 21.

But this time the scientific sweat of the ensuing decades provided a lubricant that allowed a multiplication of output almost fantastic in retrospect. The *principles* of managing enterprises of wide span and great diversity proved flexible enough to meet the test; and to allow the approach of Leadership by persuasion once again to vanquish the totalitarian, or command, bid for supremacy.

A most significant factor in this period was that the principle of "division of labor" was successfully applied to the work of Managing, itself, to entirely new degrees. The whole philosophy of Decentralization developed to new dimensions, not merely decentralization to new geographic areas and to new plants, or even decentralization to separate "product businesses" within a common corporate framework, but actually as Mr. Cordiner said in a memorable address to the American Management Association in 1945, "the decentralization of decision-making itself," so that the authority actually to *decide* was as close as possible to the work or action specifically calling for decisions.

Once more Dr. Hopf was chosen to summarize these trends. In an article called *Evolution in Organization During the Past Decade*, presented at the first post-war International Management Congress of CIOS at Stockholm in 1947, he listed the following outstanding managerial trends and advances since the seventh CIOS Congress in Washington in 1939:

1. Development of the personnel function;
2. Decentralization of management and operation;
3. Increased recognition and application of general principles of organization;
4. Creation of new units of organization to meet increasing economic and social responsibilities;
5. Improvement of techniques for policy formulation and execution.

It is also important to note at this point that the United States Management Societies were active contributors to the war effort, satisfying the insistent pleas of industry for technical aid, and, at the same time, following a process of continual measuring, tempering, and re-application of organizational and operating principles.

Such ability to handle problems—of overwhelming magnitude compared to those of the first war—and to simultaneously continue to advance the Science of Managing was due, in no small part, to the newly developing techniques of organizational communications. The engineer proved once again that the essential nature of managing was a derivative of the scientific method and that his place in guiding the affairs of men was essential to the vitality of the movement, not mere accident or prior right.

Scientists and engineers who had worked to establish a factual basis on which communications systems could be predicated, came to realize the general resemblance of large social patterns to those of specific electrical or mechanical networks. Dr. Norbert Weiner, who is widely credited with leading the study and with coining the name "Cybernetics" from the Greek word meaning the art of the pilot or steersman, said:

> One of the most interesting aspects of the world is that it may be considered to be made up of patterns. A pattern is essentially an arrangement. It is characterized by the order of the elements of which it is made, rather than by the intrinsic nature of these elements.[21]

He went on further to point out that a pattern can be used to convey information and will usually convey more information than the statement of isolated facts since it also conveys interrelations.

Dr. Weiner made the penetrating observation that there is implied in the adoption of automation as an outgrowth of Cybernetics a transcending problem, for economic and for political and social statesmen alike, in "Handling the social-political responsibility to see that some way of handling their (the permanently displaced workers) leisure is provided, to make them fit into a society that is a going concern—since, in the face of radical changes, the statesmanship of management cannot stop at the edge of the individual firm.[22]

Dr. Claude Shannon of the Bell Telephone Laboratories and others carried Weiner's structural concept into fields which provided the management scientist with working theories which would allow him to apply and measure the principles of professional management in rigorous fashion.

In the first place, they established the importance of effective communications as the key to proper operation of any system involving more than a single element. What was true of the electrical network was true in even larger measure of the corporate enterprise.

Thus, Weiner, Shannon, and their associates and contemporaries believed that "one of the lessons of Cybernetics is that any organism is held together by the possession of means for the acquisition, use, retention, and transmission of information" and that "communication of information is a problem in statistics . . . and that the theory, of course, does more than express a philosophy of communication, it provides universal measures."[23]

The ability to transmit directive information to implement change and the ability to feed back the results of the change was shown to be a function of the use of optimum channels through the minimum number of transfer or recording points. In terms of the enterprise, as an organic whole, they showed that effective operation can be only achieved when directive information, the result of decision-making, is created and applied as close as possible to the point of action, and when the channels of information, transmission, and performance feedback are soundly conceived. Thus, they further affirmed the soundness of the wide-spread decentralization of managerial authority and responsibility, so characteristic of this period, which has to an astounding extent allowed customer and public benefits from the combined social acceptability of the relatively small decentralized business and the technological and other resources of the larger modern corporation.

In the second place, the Cyberneticists showed the possibilities of mathemat-

[21] *The Human Use of Human Beings*, Dr. Norbert Weiner, Houghton Mifflin Company.
[22] Meeting of New York Chapter of the Society for Advancement of Management, 1950.
[23] "The Information Theory," Francis Bello, Fortune Magazine, December, 1953.

ical analysis in problems of business organization. Dr. Zay Jeffries, scientist and retired Vice-President of the General Electric Company, said in 1951:

> Our progress depends to a considerable extent on seeing to it that simplification processes move forward in approximate balance with the complicating processes. If this can be accomplished, then individuals with given ability can expect to go forward indefinitely without becoming casualties of their own complexity.

The simplification processes for scientific management are many, but are greatly multiplied today by the rapid emergence of essentially statistical, mathematical, and logical method. These latter kinds of developments are divorced from traditional actuarial methods by the special attitudes of mind that bring them into play. Three essential steps form the basic technique which distinguish the work:

1. Thorough exploration and precise, understandable documentation of facts concerned in an operation will reveal broad principles and affective parameters, or governing factors and variables.
2. The discovered principles and parameters may, in most instances, be defined quantatively and manipulated with predictable results, so long as the system of which they are a part is essentially stable, as the fairly mature business characteristically tends to be.
3. The disclosure of principles and the studied manipulation of affecting paraments will provide optimum procedures and processes, measurable work simplification, great precision of guidance, and better management.

This is the technique of what is now coming to be called "Operations Research and Synthesis." Thus, the mathematician provided the professional manager not only with applicable theories, but he has also provided him with an important working tool.

The general use of computers in speeding the analysis and interpretation of the operations research process is growing rapidly and is facilitating the managerial approach based on "Thinking Through" to an ever more useful degree; which incidentally but re-affirms the perceptive foresight of Taylor in taking his steel plant data, and his speeds and feeds formulas, to the mathematicians for solution even before the turn of the twentieth century.

Such use of today's modern electronics computers, moreover, progressively allows the pre-testing of an almost unlimited number of variables to determine their interrelation and their individual effect on alternative end objectives. Scientific management has thus, in effect, been handed the priceless technique of telescoping the time required in analyzing its course to an entirely new degree.

X

This "history" of the evolution of the management movement in America has, by definition, concerned itself only casually and but little with the great and important contributions of the many theorists and experimenters in the field in Europe and other parts of the World. The development of basic principles has in no sense, of course, been a purely American effort, so, in fairness, it must be pointed out that the "history" written here is thus only one American expression of what has, in fact, been a world-wide search.

CIOS, the International Committee for Scientific Management, has acted in behalf of some twenty or more member-organizations of the free world. Since its foundation in 1926, it has been a potent force in sustaining international amity through trying economic and political times. The Gold Medal of CIOS is thus widely recognized as a symbol of the highest achievement in the management field.

Another omission, necessary since this paper has dealt with the concepts of the Science of Managing, has been the catalytic activities of our own government, except when government direction has superseded normal economic interplay under the stress of global-scale War. However, the national government, representing the interests of all the people, has had the responsibility of maintaining balance among all these interests and consequently has influenced sharply, by both positive and negative stimulation, the creation of a distinctly American form of capitalism which is now enjoined by its very nature from being monopolistic.

The resultant competitive atmosphere, inherent in this philosophy of capitalism, has been one of the significant spurring forces which has helped to move Scientific Management out of the library or classroom and into the shop.

Despite all the natural forces of both politics and bureaucracy, starting with the introduction of cost cards in the Frankford Arsenal in the 1880's by Henry Metcalf and continuing through to the present day, various departments of the government have made active use of the techniques of scientific management in their own operations.

Particularly notable are, of course, the long and constructive contributions of Mr. Herbert Hoover, who, as Secretary of Commerce in the mid-twenties, made significant steps in the elimination of waste in industry and in the standardization of products; and who, with the great first President Masaryk of the then new republic of Czechoslovakia, was co-sponsor of the first International Management Congress at Prague in 1924. As President, Mr. Hoover later began a program for the improvement of governmental bureaus, and, as Chairman of the Committee on Organization of the Executive Branch of the Government, he continued this work years later during 1948 and 1949, and again in 1953 and 1954.

XI

What, then, is the present status of the Science of Managing? Is it, in the words of Mr. A. M. Lederer, the "Fifth Force," equal to and as necessary as the forces for labor, owners, government and consumers?[24]

It would seem, in practice today, impossible to deny the importance of management without denying simultaneously the factual nature and complexity of our current technological culture. In accepting the inevitable pattern which lies ahead, the place of such a Fifth Force is thus evident, and the character and

[24] Mr. Lederer, partner in the consulting firm of Morris and VanWormer, is President of the Council for International Progress in Management (U.S.A.) and Deputy President of CIOS.

quality of the Professional Manager who will both guide and discipline this "force" is of lasting moment.

In a paper written in 1951, *Notes on a Theory Of Advice*, Lyman Bryson of Columbia University emphasized the important obligation of the Professional Manager with respect to the integration of knowledge and authority which are essential counterparts of the managing function:

> The function of advice is one of the oldest in human affairs and certain abstract generalizations about it that could have been made in paleolithic times are still true.
>
> Most of these generalizations, however, have not been made and, as far as can be discovered, no standard treatise in this field has ever been written.
>
> There are mountain piles of books on salesmanship, which is not disinterested advice, and a molehill of books on leadership, but nothing on the technique and difficulties of trying to put knowledge at the service of power.
>
> The right relation of knowledge and power is, however, one of the big problems of our age.
>
> We need to give the closest scrutiny to the processes whereby decisions are made, and the effect on those decisions of rational information, if we are to master the difficulties of freedom in a time when power is so developed and knowledge is so dispersed.
>
> The function of advice is one of the crucial points in that relation and on that account may well be studied first.

Peter Drucker, in the face of such conditions, well describes the threefold job of the Manager in connection with today's business enterprise. By substituting country, institution, family, or any of the collective nouns which represent group entities for the word "company" and "enterprise," the universal nature of the "manager's" job, in this sense, is readily made apparent. As Mr. Drucker puts it:

> It is management's first responsibility to decide what economic factors and trends are likely to affect the company's future welfare.
>
> The second function of management is the organization and efficient utilization of the enterprise's human resources. In the industrial enterprise it is not individuals who produce, but a human organization.
>
> The third major function of management is to provide a functioning management. This means that management has to provide for its own succession. . . .It is tomorrow's management that will determine whether the enterprise will prosper ten years hence and indeed whether it will survive . . . today's management can at least make sure that there will be available to make tomorrow's decisions men who are fully qualified, fully trained, and fully tested in actual performance.[16]

Similar growing awareness of the necessity and almost universal applicability of the managerial functions in all areas of society has become truly international in scope. For instance, Mr. Lederer in mid-1953 described to members of the Council for International Progress in Management (U.S.A.) a renaissance of European industry which "finds its expression in a European Management

[16] *The New Society*, Peter F. Drucker, Harper & Bros., 1949, p204. See also Drucker's earlier books on *The End of Economic Man*, *The Future of Industrial Man*, *Concept of the Corporation*, and another forthcoming volume on *The Practice of Management* to be published by Harper & Bros. in the Fall of 1954.

Movement eager to catch up with a comparable management movement in the United States, which has its roots in the same philosophical belief and which has translated that belief into practical applications to an Industrial Society."[26]

The enduring place of the Scientific Management Movement, therefore, seems assured. To those who have made a professional life of its study or practice, these further words of Lawrence Appley may serve as both encouragement and credo:

> The future of America is dependent upon the caliber of management to be found in the ranks of business and industry. It is management that sets the pace and motivates labor to do its job. It is the combination of a courageous, competent management and a high-moraled, highly productive labor force that makes more things available for more people, and therefore, increases the standard of living.
>
> This management competency which is able to motivate labor to greater productivity requires sensitivity to certain moral obligations to the community. It must be understood by such management that our present form of society can be preserved only when those on the receiving end of leadership experience that for which democracy stands. If people are to know what a democracy really is, then they must enjoy its benefits in their work, as well as in their play. They must really feel and believe that their bosses are interested in them as individuals and in their development to the fullest potential of character, personality, and individual productivity.
>
> The greatest doctors, teachers, lawyers, and engineers are those who have some sense of the human values involved in their work. So it is and will continue to be with managers. The price of leadership is criticism, but its more-than-compensating reward is sense of attainment.[27]

In a hundred and fifty years we have come from narrow and dimly perceived horizons into a world of limitless possibilities and new scientific, as well as human, frontiers. The Science of Managing is, like all true sciences, creating an expanding universe of concepts and principles. Because it has come to recognize its problems as a part of, and a party to, the nature of our culture, it will continue as an unabating challenge to thought and ingenuity so long as free men continue to join in common effort to achieve desired ends.

[26] President's Report for First Six Months, CIPM, 1953.

[27] *Management and the American Future*, Lawrence A. Appley, American Management Association General Management Series, Number 169, p. 13.

INVENTORY CONTROL RESEARCH: A SURVEY*

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In past decades there have been occasional upsurges of intensive interest in inventory control problems, sometimes in the aftermath of forced inventory liquidation[1]. For the most part, the literature consisted of a few articles in business journals[2] that had but little impact on the current business behavior of the time. In the last few years we have witnessed another upsurge of interest which far surpasses any of its predecessors, with respect to the quantity and quality of the work accomplished as well as its overall impact on the business community. This vigorous research and the interest it has aroused have been made possible by parallel development in research and in business. Statisticians and economists have become interested in industrial problems concomitantly with the increased attention in business to techniques of advanced management. The appearance of this journal is an example of this harmony of interests that has come into being. The fact that modern statistical methods have been applied successfully in several instances has had much influence on the interest level of both research workers and industrialists.

The field of inventory control has proved to be a desirable field for drawing together the interests of people of widely divergent backgrounds. Because of its fundamental importance in business, management is willing to cooperate in providing descriptive data and in experimenting with the application of new theoretical models. At the same time, the problems involved have been of such a nature as to engross the attention of leading scientists in the fields of economics and statistics. The issues involved have been susceptible to approaches that run the gamut from extremely simple to highly developed abstract models.

* The author is indebted to T. V. Atwater, Jr. of M.I.T. for criticisms and comments.

[1] Lyon, Leverett S., *Hand to Mouth Buying*, Washington, D. C.: The Brookings Institution, 1929; Clark, Fred E., "An Analysis of the Causes and Results of Hand-to-Mouth Buying," *Harvard Business Review*, Vol. 6, 1927–1928, pp. 394–400; McGill, H. N., "Hand-to-Mouth Buying and Its Effect on Business," *Industrial Management*, Vol. 73, No. 6, 1927, pp. 344–347; Tosdal, H. R., "Hand-to-Mouth Buying," *Harvard Business Review*, Vol. II, 1932–33, pp. 299–306.

[2] Cooper, Benjamin, "How to Determine Economical Manufacturing Quantities," *Industrial Management*, Vol. 72, No. 4, 1926, pp. 228–233; Davis, Ralph C., "Methods of Finding Minimum-Cost Quantity in Manufacturing," *Manufacturing Industries*, Vol. 9, No. 4, 1925, pp. 353–356; Mellen, George F., "Practical Lot Quantity Formula," *Management and Administration*, Vol. 10, No. 3, 1925, p. 155; Owen, H. S., "How to Maintain Proper Inventory Control," *Industrial Management*, Vol. 69, No. 2, 1925, pp. 83–85; Owen, H. S., "The Control of Inventory through the Scientific Determination of Lot Sizes," *Industrial Management*, Vols. 70 and 71, 1925 and 1926, (nine installments). Pennington, Gordon, "Simple Formulas for Inventory Control," *Manufacturing Industries*, Vol. 13, No. 3, 1927, pp. 199–203; Wilson, R. H., and Mueller, W. A., "A New Method of Stock Control," *Harvard Business Review*, Vol. 5, 1926-1927, pp. 197–205.

Among the techniques employed in inventory control analysis are the calculus of variations, servomechanisms, decision theory, and game theory. It is encouraging that even extremely simplified formulations have found their counterpart in the world of reality and have been successfully applied in spite of (or perhaps because of) their superficiality. Considerable time must elapse before business practice catches up with the more complex models, but with a continuance of the present interchange between theoreticians and practitioners, rapid progress will be made. There is a wide scope for both research and application for many years to come.

Although much has been done toward integrating and improving several different approaches to inventory control problems, there is need for more work in this direction. Consider first the simplest type of "lot size" calculations. Formulas for calculating economical lot sizes have been in existence since 1915.[3] In its simplest form, the basic formula is derived as follows:

Assume that the expense of procurement is a constant amount for each order placed and that carrying charges may be expressed as a percentage of cost. Let Y designate the expected yearly sales (in physical units), let Q be the purchase quantity (in physical units), let C be the unit cost, and S be the procurement expense involved in making one order (in $). Then total annual variable costs (TVC) may be expressed as follows:[4]

$$TVC = \frac{QC}{2}I + \frac{Y}{Q}S$$

In order to determine the Q which minimizes total costs, we differentiate the above expression with respect to Q and set the derivative equal to zero. The following equation is obtained:

$$\frac{IC}{2} - \frac{YS}{Q^2} = 0$$

which results in the solution:

$$Q = \sqrt{\frac{2YS}{IC}} \tag{1}$$

This equation states that the economical purchase quantity varies directly with the square root of expected sales and the square root of procurement expenses and varies inversely with the square root of the carrying charges.

The problem of determining economical lot sizes in manufacturing is similar to that of finding economical purchase quantities. Here the problem is one of

[3] Ford W. Harris, "Operations and Cost," New York, 1915, pp. 48–52, referred to in F. E. Raymond, *Quantity and Economy in Manufacture*, New York, 1931, p. 121.

[4] In the absence of safety allowances, inventories vary from Q to 0. The average value of inventory is therefore $QC/2$ if the new ordering quantity arrives at the same time the old quantity is exhausted. The $QC/2$ times I represents the annual carrying charges. Y/Q represents the number of times a year that orders are placed. Therefore $(Y/Q)S$ represents the total annual procurement expenses.

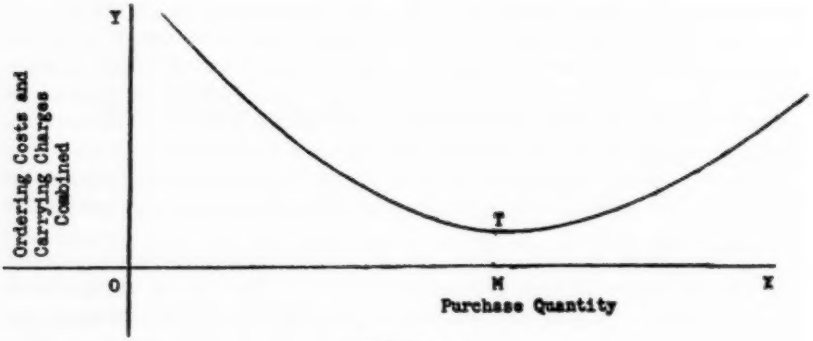

FIGURE I

balancing the initial costs for machine and clerical preparation against the costs of carrying the order in stock. If lots of smaller than optimum size are manufactured, the increase in setup costs outweighs the saving in carrying charges; if lots of larger than optimum size are manufactured, the reduction in setup costs is outweighed by the increase in carrying charges. If the letter S in formula (1) above is changed in order to designate setup costs instead of ordering costs, formula (1) as it stands may be applied to the problem of determining economic lot sizes in manufacturing. In fact, the use of formula (1) by Cleveland's Osborn Manufacturing Company resulted in an 18% saving on combined setup costs and carrying charges. The International Business Machine Corporation has been active in designing practical methods for applying this type of formula.

For those not at ease with the calculus, a graphical interpretation may be helpful. On Figure I, the horizontal axis represents the purchase quantity in physical units and the vertical axis represents the annual costs involved in ordering and in carrying this quantity. The calculus served as a tool that was helpful in finding the particular purchase quantity that minimized these costs, i.e. the quantity (OM) directly below the minimum point (T) of the curve.

Even in recent years businessmen have been "discovering" the usefulness of economical lot quantity calculations. For example, Imperial Chemical Industries, Ltd. has developed a formula equivalent to that derived above within the last five years.

More complicated analysis involving the use of lot size formulas has been carried out, allowing the inclusion of several additional factors. There is need for further work in adapting these formulas to concrete situations.

Another basic aspect of inventory control that has received much attention involves the effect of uncertainty on inventory levels. The existence of uncertainty brings about a need for safety allowances to provide protection against running out of stock because of random variations in demand. For random samples, the optimal safety allowance varies proportionately with the square root of the size of sample, that is, the safety allowance varies proportionately with the square root of mean expected demand. Research on this problem has

been carried out by T. C. Fry,[5] P. Massé;[6] C. Eisenhart;[7] K. Arrow, T. Harris, and J. Marschak;[8] and C. B. Tompkins,[9] the latter having stressed the implications of this type of analysis for military inventory control problems, in both simple and involved situations.

For purposes of exposition, consider the following example of Churchill Eisenhart. The proprietor of a store selling cigar lighters has an opportunity to replenish his stock every Monday. When average weekly demand was twenty-four lighters, computations showed that a safety allowance of twelve, that is, a total Monday stock of thirty-six lighters, was required if the proprietor was to run out only one week in a hundred; when the average weekly demand increased to one hundred units, a safety allowance of twenty-four was necessary. Thus, a quadrupling of sales was accompanied by a doubling of the safety allowance. A prevalent type of inventory control policy consists of setting safety allowances in terms of a given number of days' supply for all items. This type of policy is open to objection on the basis of purely theoretical considerations, which are also in accord with common sense considerations. Safety allowances should be larger the higher the costs of running out, the greater the uncertainty of demand, the lower the carrying charges, and the longer the delivery period, whereas the above-mentioned policy ignores all of these considerations.

The lot size and safety allowance aspects of inventory control have been combined,[10] and the interaction between them has been described.[11] However, it has been pointed out by A. Dvoretzky, J. Kiefer, and J. Wolfowitz[12] that systems based on lot sizes and safety allowances are not necessarily optimal. This paper was followed by another that indicated precisely the conditions required for their optimality.[13] Thus, there are points of contact between the high-powered statistical approach of Dvoretzky, Kiefer, and Wolfowitz, the economic and statistical approach of Arrow, Harris, and Marschak and the applications of earlier systems.

[5] *Probability and its Engineering Uses*, New York, 1928, pp. 229–232.

[6] *Les Réserves et la Régulation de l'Avenir dans la vie Economique*, Vol. I: *Avenir Determiné* Vol. II, *Avenir Aléatoire*, 1946.

[7] *Some Inventory Problems*, National Bureau of Standards, Techniques of Statistical Inference. A 2·2C, Lecture 1, January 6, 1948, Hectographed Notes.

[8] "Optimal Inventory Policy," *Econometrica*, July 1951, pp. 250–272.

[9] "Determination of a Safety Allowance," Engineering Research Associates, Inc., *Logistics Papers*, Issue No. 2, Appendix I to Bimonthly Progress Report No. 18; "Lead Time and Optimal Allowances—An Extreme Example," George Washington University, *Minutes of a Working Conference on Mathematical Problems in Logistics*, Appendix I to Quarterly Progress Report No. I, December 1949-February 1950.

[10] R. H. Wilson, "A Universal System of Stock Control," *Purchasing*, Vol. II, No. 3, 1941, pp. 80–86.

[11] T. M. Whitin, *The Theory of Inventory Management*, Princeton, 1953, pp. 56–62.

[12] "The Inventory Problem," *Econometrica*, April 1952, pp. 187–222, July 1952, pp. 450–466.

[13] "On the Optimal Character of the (s, S) Policy in Inventory Theory," *Econometrica*, October 1953, pp. 586–596.

The analysis of Arrow, Harris, and Marschak constituted a considerable extension of the previous results, including continuous dynamic analysis with

examples worked out for various functions. A year later, the results of Dvoretzky, Kiefer, and Wolfowitz appeared, these results being by far the most advanced from the standpoint of elegance, generality, and the use of high-powered mathematics. Their articles were generalized to include consideration of delivery time lags as a probability distribution, simultaneous demands for several items, interdependence of demand in the various time periods, and cases where the probability distribution of demand is not completely known.[14]

Several other techniques for analyzing inventory control problems have been formulated. Among these are linear programming[15] and servo-mechanisms.[16] The linear programming models are designed primarily for situations with important seasonal fluctuations in demand from period to period. If fluctuations in production are reduced, costs involved in overtime production are lowered, but only at the expense of increased carrying charges. The linear programming model finds the level of production for each period that minimizes combined overtime and carrying charges. Additional work is needed to relate this model to those discussed above. For example, neither lot size considerations nor random variations in demand are included in the analysis. Although these considerations are unimportant in some situations, there is a need for quantitative analysis of the differences between the results of the models.[17]

An excellent article on inventories that has received little attention because of its being written in a foreign language was published in 1938 by Erich Schneider.[18] This article considered the following problem: Given a manufacturer's sales forecast as a function of time, his initial inventory, his capacity restrictions, carrying charges, and production costs, how should he schedule his production in order to minimize costs involved in production and storage adequate to fulfill sales requirements? Schneider solved this problem for linear cost functions. The case of quadratic functions was solved by Borge Jessen in the mathematical appendix to Schneider's article. Solutions of both a graphical and analytical nature were provided. There is a strong resemblance between Schneider's results and the linear programming results discussed above. Schneider's results

[14] A description of this fundamental work that is more accessible to non-mathematicians is found in J. Laderman, S. B. Littauer and L. Weiss, "The Inventory Problem," *Journal of the American Statistical Association*, December 1953, pp. 717–732.

[15] A. Charnes, W. W. Cooper, and D. Farr, "Linear Programming and Profit Preference Scheduling for a Manufacturing Firm," *Journal of the Operations Research Society of America*, May 1953, pp. 114–129; J. F. Magee, "Production Scheduling to Meet a Sales Forecast," *Notes from M. I. T. Summer Course on Operations Research*, Cambridge, Massachusetts, 1953, pp. 134–138.

[16] Herbert A. Simon, "On the Application of Servomechanism Theory in the Study of Production Control," *Econometrica*, April 1952, pp. 247–268.

[17] Gunnar Dannerstedt of M. I. T. has constructed a model combining linear programming and an uncertain demand situation, "Production Scheduling for an Arbitrary Number of Periods Given the Sales Forecast." Program of Second Annual Meeting of Operations Research Society of America, p. 6.

[18] "Absatz, Produktion und Lagerhaltung bei einfacher Produktion," *Archiv für Mathematische Wirstschafts und Sozialforschung*, Band IV, Heft 1, Leipzig, 1938. For a review article of Schneider's work see T. M. Whitin "Erich Schneider's Inventory Control Analysis," *Joural of the Operations Research Society of America*, August 1954, pp. 329–334.

are more general than the linear programming results. Also he discusses the possibility of going beyond cost minimization. The rate of sales can be altered by price manipulation, and these alterations in the rate of sales affect costs. Schneider pointed out that the assumption that sales are evenly distributed over time is implicit in the traditional analysis of monopoly. Actually there will be a different sales function for each price policy. For each of these sales functions the optimum production function must be chosen. The expected profits for each production plan can be calculated and the price policy chosen that corresponds to the highest profit level.

The next approach to be discussed is the servomechanism approach which suggests utilizing feedback rules to adjust production to sales. For this purpose, it is suggested that a rule possessing the characteristics of a "low pass" filter is appropriate. This filter will allow production to adjust to low frequency (long range) fluctuations but will not effect changes in production for high frequency fluctuations. Servomechanism techniques can also be brought to bear on the problem of testing the stability of dynamic models. This approach has not yet been related to other approaches to inventory problems. Work is under progress at Carnegie Institute of Technology to apply servomechanism analysis to situations of uncertain demand, applying Norbert Weiner's autocorrelation methods.

A final approach to be mentioned here is the dynamic programming of Richard Bellman,[19] which makes it possible to approach problems in the calculus of variations in a new way which may be valuable from both theoretical and computational points of view. Space does not permit elaboration here.

It is interesting to note that a large proportion of the articles written on inventory control deal with only one-stage processes. Exceptions to this are the linear programming approach and the dynamic programming approach. Also the economical lot size and reorder point quantity analysis can be applied to multistage processes. Interesting complications are introduced by the additional stages. To what extent can the overall problem be decentralized and viewed as consisting of several independent stages? A quantitative answer to this problem is not presently available. In the linear programming model, the results of the "cascaded production" model can be obtained by considering the various stages separately, and perhaps more easily.[20] In models involving uncertainty there will be some deivation of the solution to the decentralized model from the optimum. The key question, which could be illuminated by some computational work, is the extent to which the solutions differ. The work of Bellman and of Dvoretzky, Kiefer, and Wolfowitz should be of great help in this connection. There is much need for exploiting available computational facilities for work on these models and on the servomechanism model. Aside from its importance in inventory control, the solution to these problems lies at the core of problems in overall systems analysis, including those of centralized versus decentralized decision-making.

A few words of caution should be inserted here to avoid the appearance of

[19] *An Introduction to the Theory of Dynamic Programming*, RAND Corporation, 1953.
[20] This is pointed out in an M. I. T. Master's Thesis by John Davidson.

overoptimism. It is not certain how much significance can be attached to the fact that fairly simple inventory control systems have been put into practice successfully, for it is not known to what extent the lack of any system that preceded their application is responsible for their success. Furthermore, there are many simple specific problems that have not yet been solved. For example, analysis of the implications of the growth in the number of products as goods move from raw materials to finished goods is only in its infancy. Similarly, existing inventory control models can be extended to incorporate in the safety allowances added inventory to provide protection against machine breakdown. This problem has far-reaching implications for the firm insofar as there is interaction between inventories of manpower and of materials. Inventories of repairmen decrease the level of safety allowance inventories of materials, as do standby workmen. Little research has been done along these lines. Although the increase in complexity over the simpler problems does not seem very great, even simple static models of this sort have not yet been completed. Also, fundamental conceptual problems concerning the data remain unsolved. What interest rate should be used in calculations of the cost of holding inventory? If the answer is the rate of return on alternative uses of funds, how is the rate to be ascertained? It is interesting to note that there is a wide divergence of viewpoint on this question. On the one hand, economists have argued for high rates, while inventory control systems have operated successfully (as evaluated by the company) using rates as low as 8% for total carrying charges. Similarly, estimates of the costs of running out of stock must be of a very crude nature, for these costs must include goodwill and the costs of idle equipment. The formulation of models in itself may perform a service by bringing about the collection of useful data required for the models.

It is encouraging to note that, in spite of these data problems, systems already in successful use involve only extremely "rough and ready" estimates. Several large corporations have found it useful to divide inventories into a few general categories of importance such as "very important," "moderately important," and "unimportant," setting safety allowances at levels corresponding roughly to the degree of importance of the item. A study at the Bell Telephone Laboratories revealed that if safety allowances are set so that the firm expects to run out of each item once a year, a $76,000 inventory is required; and if it expects to run out once very ten years, a $167,000 inventory is required.[21] Thus, even a very crude type of estimate may lead to large reductions in safety allowance levels, and hence in carrying charges.

If there is a considerable margin of error in the various parameters of a model, it is sometimes argued that the model is useless. It should be emphasized that this view is not correct. In some instances the fact that the data are always known precisely makes a model superfluous. In the case of certainly known demand, for example, many inventory problems become trivial. It is the very existence of uncertainty that makes a fairly complicated model both necessary

[21] R. H. Wilson, "A Universal System of Stock Control" *Purchasing*, March 1941, p. 85.

and desirable. Formal models are in no way intended as a substitute for human judgment. The best judgment and intuition should be incorporated in the models. The implications of inventory control systems for the economic theory of the firm have been almost completely ignored until recently. The reasons for this neglect may have been that economists have, for the most part, left uncertainty out of their analysis and have dealt with static models. It is possible to relate inventory control analysis to conventional marginal analysis. Schneider used economic analysis in his approach to the inventory problem described above. There is considerable merit to relating the two, from both the standpoint of inventory control and of economic theory. Analysis of this type helps give empirical content to the sometimes abstract curves used in economic analysis. Other authors, in suggesting that "inventories rather than marginal costs and revenues serve as a barometer to indicate when and to what extent prices or output should be changed"[22] have a mistaken notion of the relationship between inventories and economic theory. Another attempt at reconstructing economic theory on the basis of balance sheet considerations is that in Boulding's recent *A Reconstruction of Economics*. In this analysis, preferred asset ratios played a vital role, but the author spent little time explaining the basic determinants of the preferred ratios.

In addition to its importance for particular firms, inventory control analysis plays a key role in the business cycle. M. Abramovitz pointed out that during the five business cycles between the two world wars, the average changes in inventory investment were 32.4% as large as the changes in Gross National Product. Fluctuations in inventory investment were larger in terms of the value of goods involved than those of construction or durable goods, as well as being the most volatile of the main components of output.[23] Inventories play an important role in theoretical models of the business cycle.[24] In these models the distinction between desired and involuntary inventories is a difficult problem that is one of the main obstacles to be overcome. Through analysis of inventory control systems one can talk more meaningfully about this distinction. Significant progress in this direction has been accomplished.[25]

The above discussion should make clear to the reader the extensive amount

[22] W. J. Eiteman, "Price Determination, Business Practice Versus Economic Theory," *Michigan Business Reports*, No. 16, January 1949, p. 36.

[23] M. Abramovitz, *Inventories and Business Cycles*, New York, National Bureau of Economic Research, 1950, pp. 5-8.

[24] L. A. Metzler, "The Nature and Stability of Inventory Cycles," *Review of Economic Statistics*, August 1941, pp. 113-129.

[25] Ragnar Nurkse, "The Cyclical Pattern of Inventory Investment," *Quarterly Journal of Economics*, August 1952, pp. 385-408; Metzler's and Abramovitz's discussion of Abramovitz's paper at the National Bureau of Economic Research's *Conference on Business Cycles*, New York, 1951, pp. 325-333; Ruth P. Mack, "Characteristics of Inventory Investment: the Aggregate and its Parts," and Franco Modigliani, "Business Reasons for Holding Inventories and their Macro-Economic Implications," *Studies in Income* and Wealth, National Bureau of Economic Research, forthcoming; T. M. Whitin, *Theory of Inventory Management*, Princeton, 1953, pp. 109-139.

and diversity of talent that has been brought to bear on inventory problems recently. The next few years should bring about much additional research which will help decide to what extent industrial application of the various types of inventory control systems will be profitable. Contrary to the situation in the natural sciences, university research in the field of inventory control is far ahead of current practice. The prospects of consolidating the existing body of knowledge and applying it to many practical problems appear reasonably bright, as do also the prospects of advances toward improved theoretical models.

REFERENCES

Arrow, Kenneth J., Theodore Harris, and Jacob Marschak, "Optimal Inventory Policy," *Econometrica*, July 1951, pp. 250–272.

Bellman, Richard, *An Introduction to the Theory of Dynamic Programming*, RAND Corporation, 1953.

—— "Some Problems in the Theory of Dynamic Programming," *Econometrica*, January 1954, pp. 37–48.

Charnes, A., W. W. Cooper, and D. Farr, "Linear Programming and Profit Preference Scheduling for a Manufacturing Firm, "*Journal of the Operations Research Society of America*, May 1953, pp. 114–129.

Dvoretzky, A., J. Kiefer, and J. Wolfowitz, "The Inventory Problem," *Econometrica*, April 1952, pp. 187–222; July 1952, pp. 450–466.

—— "On the Optimal Character of the (s,S) Policy in Inventory Theory," *Econometrica*, October 1953, pp. 586–596.

Holt, Charles C., and Herbert A. Simon, "Optimal Decision Rules for Production and Inventory Control," (forthcoming)

Massé, Pierre, Les Réserves et la régulation de l'avenir dans la vie economique, Paris, 1946, Vol. I, Avenir determiné; Vol. II, Avenir Aléatoire.

Raymond, F. E., *Quantity and Economy in Manufacture*, New York, 1931.

Schneider, Erich, "Absatz, Produktion, und Lagerhaltung bei einfacher Produktion," *Archiv fur Mathematische Wirtschafts und Sozialforschung*, Band IV, Heft I, Leipzig 1938.

Simon, Herbert A., "On the Application of Servomechanism Theory in the Study of Production Control," *Econometrica*, April 1952, pp. 247–268.

Tompkins, C. B., "Determination of a Safety Allowance," Engineering Research Associates, Inc., *Logistics Papers*, Issue No. 2, Appendix I to Bimonthly Progress Report No. 18.

—— "Lead Time and Optimal Allowances—an Extreme Example," George Washington University, Minutes of a Working Conference on Mathematical Problems in Logistics, Appendix I to Quarterly Progress Report No. I, December 1949–February 1950.

Whitin, Thomson M., *The Theory of Inventory Management*, Princeton, 1953.

Wilson, R. H., *Scientific Control of Stockroom Inventories*, Midwest Quality Control Conference, 1947.

—— "Stockroom Purchasing," *Purchasing*, February 1951, pp. 47–50.

—— "A Universal System of Stock Control," *Purchasing*, September 1941, pp. 80–88.

ON BUS SCHEDULES*

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1. Introduction

Consider one direction of flow of buses on a network of the type shown in Fig. 1: buses of the suburban branches AD, BD, and CD merge on the trunk DF; at F they are joined by the buses of the branch EF, etc. On the basis of a traffic analysis the management of the bus company determines the amount of service required on the branches AD, BD, $\cdots$. The amount of service on a branch is specified by the number of buses running on this branch per scheduling period. To keep the following analysis flexible the scheduling period will be taken as the unit of time; its duration in terms of minutes will be introduced only at the end of the analysis.

Let N_1, N_2 and N_3 be the numbers of buses per unit time on the branches AD, BD, and CD, respectively. On the trunk DF there will then be $N_1 + N_2 + N_3$ buses per unit time. Figure 1 shows these numbers throughout the network but for one direction of traffic only, e.g. the direction of the morning rush hour traffic.

Ideally the buses should be equally spaced on each of the branches and also on the trunks. In general the given numbers N_1, N_2, $\cdots$ will not permit such an ideal solution to the *sequencing problem*. Even for a relatively simple network such as that of Fig. 1, considerable experience on the part of the scheduler will be required to achieve an acceptable compromise. In this paper a rational approach to the sequencing problem is attempted.

2. A criterion of efficiency

Consider a typical stop j on the k-th branch and assume that passengers desiring transportation in the considered direction arrive there at the fixed rate of m_{jk} passengers per unit time. For a given stop j on a given branch k and a considered scheduling period, the rate m_{jk} will be treated as a constant which is not affected by the precise schedule that has still to be worked out. It is believed that this assumption is realistic as long as the interval between successive buses is sufficiently small.

Ideally the buses on the branch k should run at equal intervals $T_k^* = 1/N_k$. Due to the conflicting demands of the other branches, however, the intervals between successive buses on the branch k will not be equal to each other. The variation of the number of waiting passengers with time is then shown by the heavy line in Fig. 2. The area between this line and the time axis represents the amount W_{jk} of man hours spent in waiting at the stop j per hour.

* The results presented in this paper were obtained in the course of research sponsored by the International Business Machines Corporation of New York City.

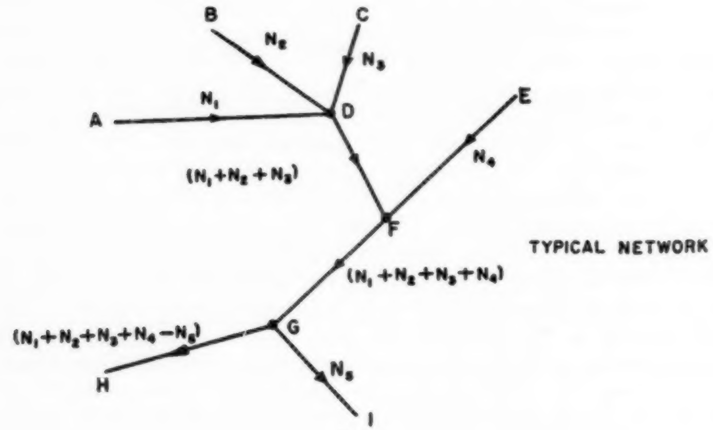

FIG. 1

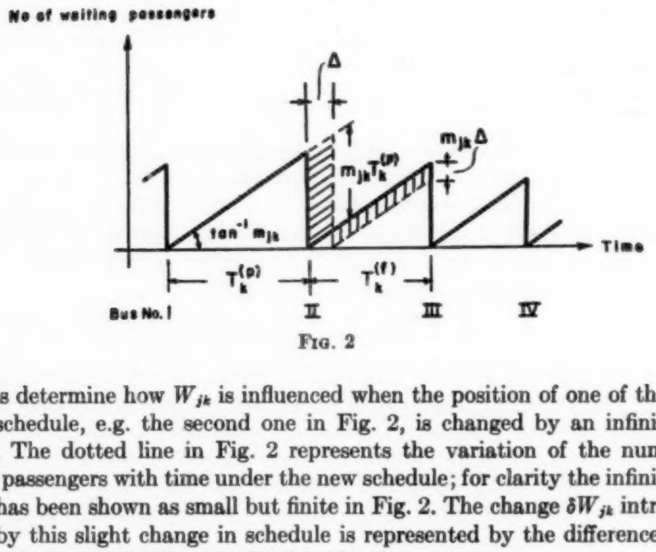

FIG. 2

Let us determine how W_{jk} is influenced when the position of one of the buses in the schedule, e.g. the second one in Fig. 2, is changed by an infinitesimal time Δ. The dotted line in Fig. 2 represents the variation of the number of waiting passengers with time under the new schedule; for clarity the infinitesimal time Δ has been shown as small but finite in Fig. 2. The change δW_{jk} introduced in W_{jk} by this slight change in schedule is represented by the difference of the cross-hatched areas in Fig. 2 and is found to be

$$\delta W_{jk} = m_{jk}[T_k^{(p)} - T_k^{(f)}]\Delta, \tag{2.1}$$

where a term in Δ^2 has been neglected and $T_k^{(p)}$ and $T_k^{(f)}$ are the time intervals between the considered bus and the preceding and following buses.

The total amount W_k of man hours spent in waiting by passengers on the branch k per hour is

$$W_k = \sum_j W_{jk}. \tag{2.2}$$

By the considered change in schedule W_k is changed by

$$\delta W_k = \sum_j \delta W_{jk}. \tag{2.3}$$

Since the time spacings $T_k^{(p)}$ and $T_k^{(f)}$ are the same for all stops on the branch k, we have from (2.1) and (2.3)

$$\delta W_k = [T_k^{(p)} - T_k^{(f)}]\,\Delta \sum_j m_{jk}. \tag{2.4}$$

Here the sum represents the total number of passengers per unit of time desiring transportation along the branch k in the considered direction. This sum must therefore bear some relation to N_k, the number of buses assigned to this branch. We shall set

$$\sum_j m_{jk} = \alpha_k N_k, \tag{2.5}$$

where α_k is a factor indicating the importance of the branch k in the network. The absolute values of the *important factors* attributed to the various branches are irrelevant, only their ratios matter; in other words the importance factor of one of the branches, e.g. the trunk with the heaviest concentration of buses, may arbitrarily be assigned the value one. In many cases it will be appropriate to assign an importance factor of unity to every branch in the network.

In the following, a bus schedule will be called *efficient* if it minimizes the amount of man hours spent in waiting per hour by the passengers throughout the network. The amount to be minimized is

$$W = \sum_k W_k, \tag{2.6}$$

where the summation must be extended over every branch and trunk of the network. For an efficient schedule, W must not be changed if any one bus is scheduled an infinitesimal amount Δ later than originally intended. According to (2.4) and (2.5) an efficient schedule must therefore satisfy equations of the form

$$\sum_k \alpha_k N_k [T_k^{(p)} - T_k^{(f)}] = 0, \tag{2.7}$$

where the summation includes all branches and trunks of the network. The time intervals in the parenthesis of (2.7) are those preceding and following the appearance of the considered bus on each of the branches and trunks. An equation of the type (2.7) is associated with each bus in the schedule. An additional equation is obtained from the fact that the time intervals on the most heavily travelled trunk must add up to the scheduling period, i.e. to the unit of time. When this condition is fulfilled for this trunk, the corresponding conditions for all other branches or trunks are automatically satisfied.

The manner in which Eqs. (2.7) are conveniently established is best explained in connection with typical examples.

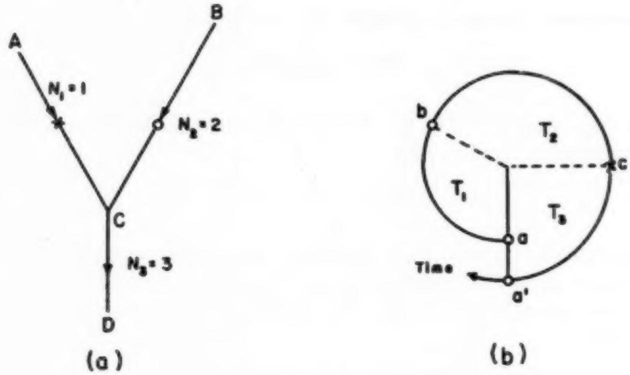

FIG. 3

3. Examples

I. Consider the simple Y network of Fig. 3(a). The branch AC has one bus per unit of time, the branch BC two buses and the trunk CD three buses. Assume that the importance factor unity is assigned to each branch and the trunk.

The schedule for the trunk CD can be represented on a spiral diagram (Fig. 3(b)): a complete, clockwise turn on the spiral represents one scheduling period; the buses a and b marked by small circles come from the branch BC, and the bus c marked by a cross comes from the branch AC; the intervals of time between the buses on the trunk are denoted by T_1, T_2, and T_3, so that

$$T_1 + T_2 + T_3 = 1. \tag{3.1}$$

The first bus a and the corresponding bus a' at the beginning of the next scheduling period may be considered as fixed, and time shifts Δ need to be applied only to the buses b and c. On the branch BC, the time intervals preceding and following the appearance of the bus b are T_1 and $T_2 + T_3$; on the trunk CD, however, the time intervals preceding and following the appearance of the bus b are T_1 and T_2. For the bus b Eq. (2.7) therefore takes the form

$$3(T_1 - T_2) + 2(T_1 - T_2 - T_3) = 0. \tag{3.2}$$

Similarly, for bus c, we have

$$3(T_2 - T_3) = 0, \tag{3.3}$$

because on the branch AC the time intervals preceding and following the appearance of this bus are both equal to the full scheduling period so that their difference is zero.

The solutions of Eqs. (3.1) through (3.3) are

$$T_1 = \frac{7}{17}, \qquad T_2 = T_3 = \frac{5}{17}. \tag{3.4}$$

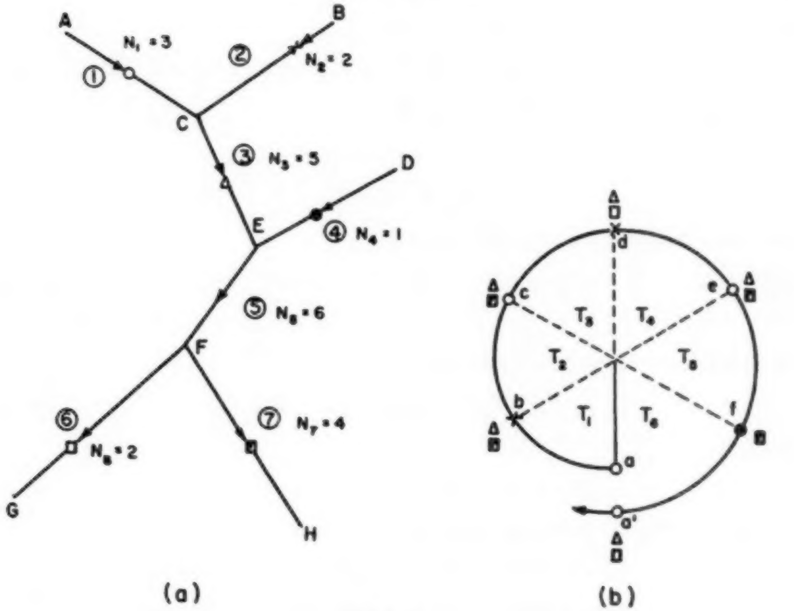

FIG. 4

If the unit of time, i.e. the scheduling period, is fifteen minutes, these figures are equivalent to

$$T_1 = 6.2 \text{ min}, \qquad T_2 = T_3 = 4.4 \text{ min}. \tag{3.5}$$

By rounding off these figures to the nearest half-minute we obtain the following schedule:

on the branch AC: buses run every 15 min,
on the branch BC: buses run alternatingly at 6 and 9 min
and on the trunk CD: the time spacings are 6, 4.5, and 4.5 min.

II. Figure 4(a) shows a somewhat more realistic example of a network. The various branches and trunks are labelled (1) to (7) and the desired numbers of buses per unit of time are indicated on each branch. Except for the busiest trunk (5), each branch or trunk is given a symbol (hollow or full circle, cross, triangle, etc.) to identify the buses running on it. Figure 4(b) shows an assumed spiral diagram for the trunk (5); the symbols appearing on this diagram should be interpreted with the help of Fig. 4(a). Assuming that all branches and trunks have the same importance factor and applying (2.7) we obtain:

for bus b, branches BC, CE, EF and FH:

$$\begin{aligned} 2(T_1 + T_6 + T_5 + T_4 - T_2 - T_3) + 5(T_1 - T_2) + 6(T_1 - T_2) \\ + 4(T_1 + T_6 - T_2) = 0 \end{aligned} \tag{3.6}$$

for bus c, branches AC, CE, EF and FH:

$$3(T_1 + T_2 - T_3 - T_4) + (5 + 6)(T_2 - T_3) + 4(T_2 - T_3 - T_4) = 0 \tag{3.7}$$

for bus d, branches BC, CE, EF and FG:

$$\begin{aligned} 2(T_2 + T_3 - T_4 - T_5 - T_6 - T_1) + (5 + 6)(T_3 - T_4) \\ + 2(T_1 + T_2 + T_3 - T_4 - T_5 - T_6) = 0 \end{aligned} \tag{3.8}$$

for bus e, branches AC, CE, EF and FH:

$$\begin{aligned} 2(T_3 + T_4 - T_5 - T_6) + 5(T_4 - T_5 - T_6) + 6(T_4 - T_5) \\ + 4(T_3 + T_4 - T_5) = 0 \end{aligned} \tag{3.9}$$

for bus f, branches DE, EF and FH:

$$6(T_5 - T_6) + 4(T_5 - T_1 - T_6) = 0 \tag{3.10}$$

and

$$T_1 + T_2 + T_3 + T_4 + T_5 + T_6 = 1. \tag{3.11}$$

The solutions of these equations are

$$\begin{aligned} T_1 = 0.1556, \qquad T_2 = 0.2148, \qquad T_3 = 0.1791, \\ T_4 = 0.1585, \qquad T_5 = 0.1771 \text{ and } T_6 = 0.1149. \end{aligned} \tag{3.12}$$

If the scheduling period is fifteen minutes, the values (3.12) rounded to the nearest half minute give the following table of time intervals:

Branch 1: (5.5) – (5) – (4.5)
Branch 2: (5.5) – (9.5)
Branch 3: (2.5) – (3) – (2.5) – (2.5) – (4.5)
Branch 4: (15)
Branch 5: (2.5) – (3) – (2.5) – (2.5) – (2.5) – (2)
Branch 6: (8) – (7)
Branch 7: (3) – (5) – (2.5) – (4.5).

III. Consider the divided circuit of Fig. 5(a). The buses start on the branch (1), divide unequally into the branches (2) and (3), and reunite on the branch (4). The N_k and α_k are marked on the figure.

Whereas in the previous examples the travel times of the buses did not enter into the picture, the travel times on the branches (2) and (3) are now important. Let us suppose that it takes D units of time more to travel from P to Q on (2) than on (3).

In Fig. 5(b), the inner spiral represents the buses on the branch (1) and the outer spiral the buses on the branch (4). Applying Eq. (2.7) to each bus in turn, we obtain:

for bus b:

$$4(T_1 - T_2) + 2(T_1 - T_2 - T_3) + 3(T_1 - T_2 + D) = 0, \tag{3.14}$$

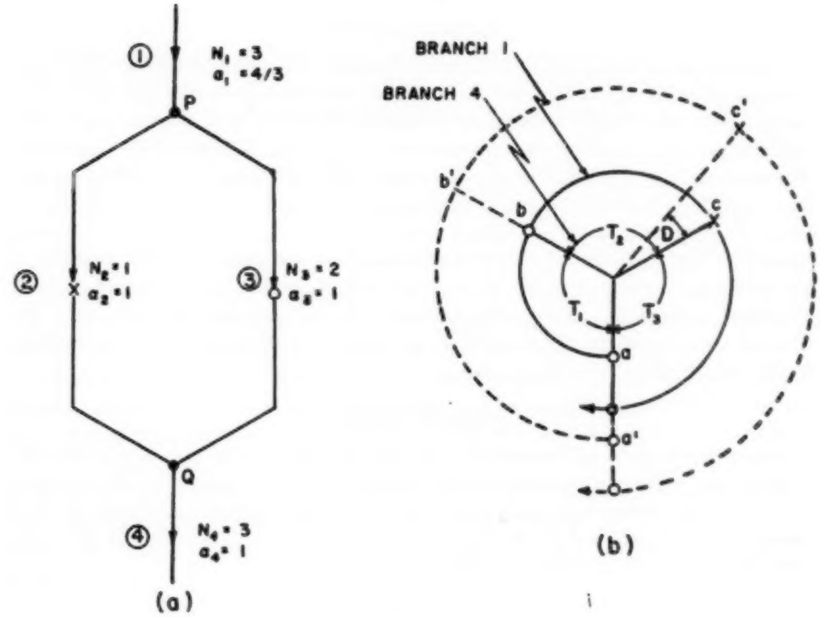

Fig. 5

for bus c:

$$4(T_2 - T_3) + 3(T_2 - D - T_3 + D) = 0. \tag{3.15}$$

Moreover,

$$T_1 + T_2 + T_3 = 1. \tag{3.16}$$

The solution of Eqs. (3.14) through (3.16) is:

$$T_1 = .380 - .207D, \qquad T_2 = T_3 = .310 + .103D. \tag{3.17}$$

Let the scheduling period be 15 min and assume that $D = 1/15$, i.e. that the travel time on the branch (2) exceeds that on (3) by 1 min. Equations (3.17) then furnish the following values in min, rounded to the nearest half minute:

$$T_1 = 5.5, \qquad T_2 = T_3 = 4.5. \tag{3.18}$$

The time intervals on the various branches are:

Branch 1: 5.5 – 4.5 – 4.5
Branch 2: 15
Branch 3: 5.5 – 9
Branch 4: 5.5 – 3.5 – 5.5.

4. Conclusions

The equations which were used in the examples of the preceding section were formulated with the aid of a spiral diagram which represents the result of a preliminary scheduling in so far as it assigns each bus to a definite route and indicates how the various branches are geared together. In complicated networks the construction of a suitable spiral diagram may itself become a difficult problem, particularly when the buses flowing into a main branch (e.g. FG in Fig. 1) have to be geared with the buses flowing out of this branch. The following procedure has been found useful in this case. Of the possible arrangements of the buses flowing into and out of the main branch, select the two which present the most symmetric appearance; perform the minimizing calculation on the various ways of fitting these patterns together. The optimum schedule is likely to be one of these combinations.

Acknowledgement

The authors were introduced to the problem treated in this report through several inter-office memoranda of the Capital Transit Company of Washington, D. C. and the International Business Machines Corporation. A discussion with Messrs. F. G. Awalt and W. H. Nairn of CTC and Dr. G. W. Petrie of IBM was particularly helpful in this connection.

THE STEPPING STONE METHOD OF EXPLAINING LINEAR PROGRAMMING CALCULATIONS IN TRANSPORTATION PROBLEMS*

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1. Introduction

Recent scientific research has made available a variety of tools which can be brought to bear on managerial problems. One of these tools, which has already proved its worth in practical applications, is linear programming.

It is not the purpose of this article to explore the full ramifications of these research developments or even to explore the general field of linear programming. Only transportation-type problems and models will be discussed in any detail.

Transportation-type problems have certain features which makes it possible to devise special computational techniques which are extremely simple to understand and apply. The illustrative example to be presented in this paper will be employed to explain the "stepping stone" method for solving these kinds of problems.[1]

Succeeding discussion will undertake to illuminate the nature of transportation-type problems and the kinds of situations and questions to which these methods may be applied. An effort will be made to keep the discussion on an elementary level. Readers interested in more technical expositions or in treatments of the general field of linear programming may refer to the appendix or to the references cited in the bibliography.

2. A Transportation Example

Managerial decisions are directed toward choosing between potential courses of action in order to secure the best possible outcome, as measured by costs, profits, or some other suitable criterion. While numerous alternatives may be available, the level of achievement is ordinarily limited by the necessity of meeting certain prescribed conditions. A traffic manager may desire, for example, to schedule freight shipments in a manner which will insure the movement of goods at lowest total cost. He does not, however, operate with a "clean slate."

* This paper was written as part of the project, "The Planning and Control of Industrial Operations" under grant from the Office of Naval Research. It represents a revised version of a paper originally presented at the Case Institute of Technology and included in their *Proceedings of the Conference on Operations Research in Marketing*, Cleveland, 1953.

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[1] This method is relatively easy to explain. An alternate method developed by Dantzig [4] has certain advantages for large-scale hand calculations. However, the authors have generally found it desirable to use a combination of both methods in such calculations as they have dealt with.

The goods are usually available at more or less convenient points for delivery to prescribed destinations. Freight schedules which do not meet these conditions are not regarded as suitable. Ordinarily, it is possible to devise many different schedules which meet these conditions at varying levels of total costs. The traffic manager is therefore concerned with devising some method for selecting the least costly of such suitable schedules.

The aspect of linear programming which is concerned with optimizing a criterion (such as total costs) while meeting certain prescribed conditions (such as required delivery schedules) provides a method of attack for these types of problems. Moreover, the types of problems which this traffic manager confronts possess a special kind of mathematical structure by means of which the data may be set forth in a tabular array in terms of which the best (or least costly) solution may be obtained in a straightforward manner.

Tables IA and B provide a simple hypothetical example of such a problem. In this example the traffic manager is required to ship 16 units (e.g., tons) of a homogeneous good to 5 destinations—D_1, D_2, D_3, D_4 and D_5. The amount to be delivered at each destination is listed in the bottom row of Table IB. Thus, within the time interval being considered, 2 units are required at D_1, 2 at D_2, etc. Moreover, the 16 units of goods to be shipped are available in the amounts listed at 3 different origins: 5 units are available at origin O_1, 5 units at O_2 and 6 units at O_3.

The "rims" of Table IB thus portray the number of units available and required. The numbers in the right-hand *column* represent the amounts available

TABLE I

A. Unit Shipment Costs

Origins	Destinations				
	D_1	D_2	D_3	D_4	D_5
O_1	c_{11} −2	c_{12} −1	c_{13} −2	c_{14} −3	c_{15} −3
O_2	c_{21} −2	c_{22} −2	c_{23} −2	c_{24} −1	c_{25} +1
O_3	c_{31} −3	c_{32} −3	c_{33} −2	c_{34} −1	c_{35} −2

B. Physical Program Requirements

Origins	Destinations					
	D_1	D_2	D_3	D_4	D_5	Totals
O_1	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	5
O_2	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	5
O_3	x_{31}	x_{32}	x_{33}	x_{34}	x_{35}	6
Total.....	2	2	4	4	4	16

at each origin and the numbers in the bottom *row* represent the amounts required at each destination. In this case the sum total of the amounts available (16 units) is equal to the sum total of the amounts required.

The intersection of each row and column represents a possible routing. The value of x_{11}, inserted at the intersection of the first row and first column, represents the amount scheduled for shipment from O_1 to D_1. Similarly, x_{12} and x_{21} represent, respectively, the amounts to be shipped from O_1 to D_2 and from O_2 to D_1. In general, then, x_{ij} represents the amount to be scheduled from the i^{th} origin, O_i, to the j^{th} destination, D_j. The first subscript ($i = 1, 2, 3$) designates the origin and the second ($j = 1, 2, 3, 4, 5$) designates the destination. The numerical value (to be determined) for each variable or unknown, x, represents the amount to be shipped from the origin indicated by the first subscript to the destination indicated by the second.

Table IB portrays only physical aspects of the problem—i.e., the number of units of goods available and required and the routes by which shipments may be effected. There are, however, numerous ways by which these physical requirements may be fulfilled. A possible criterion for choosing among these various solutions is supplied by the total cost of shipping the indicated quantities. The unit costs necessary to make these determinations are set forth in Table IA. Thus, $c_{11} = \$2$ is the cost per unit of shipping goods from O_1 to D_1; $c_{12} = \$1$ and $c_{21} = \$2$ are, respectively, the unit costs of shipment from O_1 to D_2 and from O_2 to D_1. Generally, c_{ij} represents the unit cost of shipment from the i^{th} origin, O_i, to the j^{th} destination, D_j.[2]

It is clear that costs will increase in proportion to the amounts shipped over any route. Total costs are determined by multiplying the amounts scheduled for shipment over each route by the appropriate freight rates and summing the results of these multiplications. These two properties (known, mathematically, as "homogeneity" and "additivity") classify the cost function as "linear." These same linearity properties also apply to the restrictions. The amounts available and required are reduced in proportion to the amounts scheduled for shipment at each origin and destination. Moreover, the total amounts shipped and received are obtained by summing across, respectively, the rows and columns of Table IB.

If the objective is to determine the best (or least costly) program, the problem of linear programming and the problem of the traffic manager are identical. The former is concerned with optimizing a linear form subject to a set of linear

[2] Thus $c_{24} = -1$ and $c_{25} = +1$ represent the cost per unit shipped from O_2 to D_4 and to D_5, respectively. Most of the values are entered with a negative sign. Had this been a *maximization* problem—e.g., of money receipts or profits—the reverse sign convention would have been employed. There is a mathematical reason for this which need not be detailed here since it is explained in the Appendix. The positive value for c_{25} is entered primarily to illustrate that the analysis holds even when some values are negative and some positive. It would also hold if values $c_{ij} = 0$ had been entered in the table. For interpretive purposes the negative values may be thought of as costs and the positive value as some sort of unit receipt—e.g., a subsidy—above costs. A value of zero might indicate that a particular origin and destination were identical—and so on, the interpretation being altered, as required, for each problem.

restrictions. The latter is concerned with fulfilling the restrictions set forth in Table IB in the least costly manner, as determined by reference to the rates set forth in Table IA.

3. A Method of Obtaining a First Solution

A trial solution for the x_{ij}, showing the amounts to be shipped from the i^{th} origin to the j^{th} destination, may, as a matter of routine, be obtained by applying "the northwest corner rule":

The Northwest Corner Rule[3]

1. Start in the northwest corner of Table IB and compare the amount available at O_1 with the amount required at D_1.
2. Set $x_{11} = 2$, the *smaller* of these two values. This satisfies requirements at the first destination without drawing away more than is available at the first origin. In fact, 3 units are still available at the first origin.
3. Move the next step away from the northwest corner to the square marked x_{12}.[4] Compare the amount still available, 3 units, at O_1 with the amount required, 2 units, at D_2. Write in the smaller of the two values so that $x_{12} = 2$.
4. Move the next step away from the northwest corner to the cell marked x_{13}. There is only 1 unit now remaining at O_1 and 4 units required at D_3. The smaller of these two values $x_{13} = 1$ is entered in this cell, finally exhausting the amount available at O_1 (by shipments to D_1, D_2, and D_3, respectively) but leaving a requirement of 3 units still to be filled at D_3.
5. Move the next step away from the northwest corner to the cell marked x_{23} and enter the smaller of the two amounts: $x_{23} = 3$.
6. Continue in this manner outward, step by step, from the northwest corner until, finally, a value $x_{35} \geq 0$ is entered in the southeast corner.[5]

The circled values in Table II show the solution obtained in this manner. The part of the program that applies to the 5 units available at O_1 is as follows: 2 units to D_1; 2 units to D_2; and 1 unit to D_3—exhausting the 5 units available at O_1 and fully satisfying requirements at D_1 and D_2. Hence, 5 units remain at O_2 and 6 units at O_3 to satisfy remaining requirements at D_3, D_4, and D_5. These are to be distributed as follows: From O_2, 3 units are to be shipped to D_3 and 2 units to D_4. From O_3, 2 units are to be shipped to D_4 and 4 to D_5. That the circled values constitute a solution—i.e., satisfy the rim requirements—may be verified by adding the circled values to obtain the total for each row or for each column. Cells which do not contain circled values represent $x_{ij} = 0$. That is, zero units are scheduled to be shipped from these origins to these destinations. The total program cost is determined by multiplying each x_{ij} by its correspond-

[3] The essentials of these rules were developed by G. B. Dantzig and his associates at the U. S. Department of the Air Forces, Project SCOOP—Scientific Computation of Optimum Programs. See [4].

[4] Had the amount required at D_1 exceeded the amount available at O_1 the latter value would have been written into the square for x_{11}. The next step from the northwest corner would have been to the cell marked x_{21}. Had the amount available at O_1 been *equal* to the amount required at D_1 the next step away from the northwest corner would have been to the cell marked x_{22}.

[5] The symbol $(\geq)$ means that the value on the left is greater than or equal to the value on the right. The symbol $(\leq)$ reverses this relation.

TABLE II

First "Basic" Program[a]

N. W. →

↓

Origins	Destinations					
	D_1	D_2	D_3	D_4	D_5	Total
O_1	2	2	1	2	1	5
O_2	0	1	3	2	−3	5
O_3	1	2	0	2	4	6
Total.........	2	2	4	4	4	16

$$\text{Total Cost}\begin{cases} x_{11}c_{11} + x_{12}c_{12} + x_{13}c_{13} + x_{23}c_{23} + x_{24}c_{24} + x_{34}c_{34} + x_{35}c_{35} = \Sigma_{i,j}\, x_{ij}c_{ij} \\ 2(-2) + 2(-1) + 1(-2) + 3(-2) + 2(-1) + 2(-1) + 4(-2) = -26. \end{cases}$$

ing c_{ij} (from Table IA) and summing. At the bottom of Table II where this operation has been carried out, the total cost is shown to be $26.

4. A Test for Determining Whether a Best Solution Has Been Obtained

A question now arises as to whether this program is "best," or whether a different program is possible which also satisfies the stipulations but at a lower total cost. This may be determined in the following manner:

1. Choose a cell in which no circled item appears and determine a non-circled number with which to "evaluate" this cell relative to the present program as follows:

2. Choose a circled value in the *same row* as the cell to be evaluated. Choose this circled value in a way which will admit of a step (or jump) to another circled value in the *same column*.

3. In this manner movement is effected from a circled item in one row to a circled item in another row. From the latter position move to a circled item in the same *row*.

4. Continue in this manner moving alternately first in a column and then in a row until the same *column* is reached for which a non-circled value is sought. In other words, beginning on a circled item in the same *row* for which the non-circled value is sought, end up on a circled value in the same *column* for which the non-circled value is sought.

5. Locate the c_{ij} which occupy the same cells as the circled items used in moving from the beginning to the end of the path. Attach a plus sign to the c_{ij} associated with the beginning cell of the path; attach a minus sign to the c_{ij} associated with the cell occupied after the first step was taken. Continue in this manner reading the c_{ij} from Table IA and attaching alternate plus and minus signs until all the steps have been accounted for. Form the sum of these plus

[a] Any of more than one (i.e., equal) most negative values may be arbitrarily chosen as the indicator.

and minus values and then subtract the c_{ij} associated with the cell for which the non-circled value is sought.

6. Repeat the process until all cells not occupied by circled items are evaluated. The appearance of *only* non-negative values (zero or greater) indicates that a best possible program has been reached. When this is not the case (see the value underlined at O_2-D_5 of Table II) improvement is possible.

5. An Illustrative Calculation

Tables II and IA may be used to provide concrete illustrations. Suppose that it is desired to determine the non-circled value at O_2-D_5 in Table II (Instruction 1). The steps are as follows: Begin on the circle at O_2-D_4 (Instruction 2). Move to the circle at O_3-D_4 (Instruction 2). End at the circle O_3-D_5 (Instructions 3 and 4). Refer to Table IA and select the unit costs (Instruction 5). At O_2-D_4 the unit cost is $c_{24} = -1$; at O_3-D_4 the cost is $c_{34} = -1$; at O_3-D_5 the unit cost is $c_{35} = -2$. To these costs are assigned alternate plus and minus signs to yield the sum $+c_{24} - c_{34} + c_{35} = +(-1) - (-1) + (-2) = -2$ (Instruction 5). From this total is deducted the unit cost $c_{25} = +1$ appearing in the cell $O_2 - D_5$ whose non-circled value is being sought (Instruction 5). The result, -3, appears underlined at O_2-D_5 in Table II. Since this value is negative it may be pos sible to locate a less costly program (Instruction 6).

6. An Interpretation of the Procedure

Before proceeding to show how the solution of Table II may be modified in order to obtain an optimum solution, certain points may be emphasized. As values for the x_{ij} are obtained they should be circled in order to distinguish them from other numbers which may be entered in the table. Having obtained these seven values, the next step is to explore all parts of the region surrounding the solution. The whole table may be regarded as a pond in which stepping stones (the circled values) have been placed in certain positions. The explorer is permitted to step or jump from stone to stone in his exploration of the pond in much the same manner as the rules governing a rook's (or castle's) moves in chess. He must begin and end each move on a stone provided by the solution. When alternate stones are available along the route he should choose the shortest path from the beginning *row* to the ending *column*. In most cases this condition will be satisfied by (1) choosing the initial *row* stone closest to the cell being evaluated but (2) so situated that the explorer will move out of this row on the first move and (3) making as few moves as possible reach the stone nearest his objective in the ending *column*.

7. Procedure for Obtaining Another Solution

Essentially, exploration is being conducted in order to evaluate (in cost terms) the desirability of moving one of the stones to a new position. Whenever a negative value appears it is desirable to move *one* of the stones from its current position to the cell in which the negative (uncircled) value appears. If more than one negative value appears, the *most* negative one is chosen to indicate the spot where the greatest cost reduction for program accomplishment is promised.[6]

TABLE III

A. Diagram of Evaluation Paths

Origins	Destinations					
	D_1	D_2	D_3	D_4	D_5	Totals
O_1	⊕	⊖	⊖			5
O_2			⊕	×	②	5
O_3				○	⊕	6
Totals.......	2	2	4	4	4	16

Note: The solid arrows indicate the path which would be used to evaluate $O_3 - D_3$; the broken arrows indicate how this path would be continued to evaluate $O_3 - D_1$. Plus and minus signs are indicated for each c_{ij} (from Table IA) in arriving at these two evaluations. These values, after subtracting c_{33} and c_{31}, respectively, are shown in Table IIIB, as now circled amounts.

B. Second Basic Solution

Origins	Destinations					
	D_1	D_2	D_3	D_4	D_5	Totals
O_1	②	②	①	5	4	5
O_2	0	1	③	5	②	5
O_3	−2	−1	−3	④	②	6
Totals.......	2	2	4	4	4	16

A question remains as to which stone to move. The rule is as follows: (i) Retrace the path used to explore the cell in which the most negative value appeared. (ii) Select those stones which were assigned a plus value in the alternation between plus and minus signs used in evaluating these cells. (iii) From these positively marked stones select the one with the *smallest* value written in its circle.[7] (iv) Form a new table. (v) Move this stone (at the indicated smallest numerical value) from its position to the cell previously occupied by the most negative value. (vi) Finally, enter all of the other stones at their previous position but *without* any numbers inside these circles.

8. Illustrative Calculation

The instructions to this point may be illustrated by constructing Table IIIA from Table II. The most negative value is -3 and is so indicated by the underline drawn in cell O_3-D_3 of Table II. In arriving at this evaluation the steps provided by the solution from Table II were $+(O_2 - D_4)$, $-(O_3 - D_4)$,

[7] This rule is necessary to avoid the appearance of negative shipments—i.e., shipments *from* destinations to origins. See remark in footnote 6 in case more than one smallest positive value occurs.

$+(O_2 - D_5)$. Only the first and third steps had positive signs assigned to their related c_{ij}. The smaller of the two positive stones was ②. This value is, therefore, entered where $-\underline{3}$ appeared in Table II. The other circles (in blank) are then entered in the same position as before as a comparison between Tables II and IIIA reveals. The × in Table IIIA indicates the position of previous location of ②.

A new solution is now obtained by filling the circles (with reference to the rim stipulations) and thus securing a new set of stones. The new program so obtained is indicated by the circled values of Table IIIB.

9. Evaluation and Continuation

Multiplication of the circled values by their appropriate unit costs (from Table IA) shows that movement from Table II to IIIB has effected a reduction in program cost from $26 (Table II) to $20 (Table IIIB). But evaluation of this new program results again in negative (non-circled) values. Still further improvement is, therefore, possible. Among the three negative values the most negative one is -3, as indicated by the underline at O_3-D_5 in Table IIIB. A stone is therefore to be moved into this cell. Retracing the path and noting the steps with *positive* signs (see note at bottom of Table IIIA) the first stone ② in the path is designated for movement to the new position. As before the other circles are first entered in blank at their previous position and then "filled up" by reference to the rim stipulations.

The new program is displayed in Table IV. The total program cost is $14 so that a further reduction of $6 has been made in proceeding from Table IIIB to Table IV. Moreover, the evaluation procedure results in no negative (uncircled) values in any of the cells so that the program is "best." No further cost reductions are possible.

10. Alternate Best Programs

The zero at O_2-D_4 of Table IV has been underlined for a particular reason. In many problems (such as the present one) equally good lowest cost, or best, programs may be available. The ability to discover such programs increases the range of choice and adds flexibility to the sharpness of the tools that

TABLE IV

Third Basic (Best) Program

Origins	Destinations					
	D_1	D_2	D_3	D_4	D_5	Totals
O_1	②	②	①	2	4	5
O_2	0	1	①	$\underline{0}$	④	5
O_3	1	2	②	④	3	6
Totals.......	2	2	4	4	4	16

TABLE V

Fourth Basic (Best) Program

Origins	Destinations					
	D_1	D_2	D_3	D_4	D_5	Totals
O_1	2	2	1	2	4	5
O_2	0	1	0	1	4	5
O_3	1	2	3	3	3	6
Totals.......	2	2	4	4	4	16

linear programming provide. It allows the user of these tools to present, for managerial consideration, a wide range of equally good choices and by examining the bases on which such choices are made, it provides him with an opportunity for discovering suppressed criteria. Discovery of such criteria allows him to sharpen his instruments for the next application.

11. Procedure for Discovering Additional Best Programs

The methods previously described may be extended to the location of such programs.[8] After (but only after) one such best program has been reached the (non-circled) zeros may be treated in the same fashion as the negative values were previously treated. Outside of this switch in attention from negative to zero evaluation cells the procedure is precisely the same as before.

Selecting the 0 at O_2-D_4 in Table IV and retracing the steps by which this evaluation was made, the value 1 is indicated for movement from O_2-D_3 to O_2-D_4. Table V displays the alternate best program which results from this adjustment. Its cost is $14, precisely the same as the program given in Table IV.

The zero at O_2-D_3 in Table V is at precisely the point where the switch of stones was made in proceeding from Tables IV to V. This zero may, therefore, be ignored because reswitching would only lead back to Table IV. That is, just as the zero of Table IV indicated the existence of another best program (Table V) so Table V indicates that the procedure can be reversed to give Table IV. However, both tables have a zero at O_2-D_1. Hence, another equally best solution is available. Using Table V and switching 1 from O_2-D_4 to O_2-D_1, the program of Table VI is derived. The program cost is, again, $14.

Inspection of Tables IV, V, and VI reveals that no new zeros have appeared. All possible (basic) best programs have, therefore, been discovered.

12. Derived and Basic Programs

The reason for the constant reference to "basic" programs may now be indicated. So long as only integers (whole numbers) appear along the rims, no frac-

[8] They may also be used to locate second best, third best, solutions, and so on. Where management is considering such choices, the methods of linear programming also provide, simultaneously, the associated (opportunity) costs.

TABLE VI

Fifth Basic (Best) Solution

Origins	Destinations					
	D_1	D_2	D_3	D_4	D_5	Totals
O_1	1	2	2	2	4	5
O_2	1	1	0	0	4	5
O_3	1	2	2	4	5	6
Totals.......	2	2	4	4	4	16

TABLE VII

Sixth Derived (Best) Program

Origins	Destinations					
	D_1	D_2	D_3	D_4	D_5	Totals
O_1	5/4	2	7/4			5
O_2	3/4			1/4	4	5
O_3			9/4	15/4		6
Totals.......	2	2	4	4	4	16

tions will enter as program elements in any of the cells. This happens to be a mathematical property of the method of calculation employed. But if fractional values are admitted, then, where more than one basic optimal program exists, a host[9] of additional equally best programs become possible.

The method of obtaining one of these fractional programs may be briefly set forth. Any two (non-negative) fractions whose sum is one may be applied to any two of the best program tables to derive one of these additional programs.[10] All cells containing non-circled items represent $x_{ij} = 0$. The chosen fractions are applied, cell by cell, to each table and the resulting proportions of each (previously obtained) program combined to form the cells of a new best program.

This discussion may be made more concrete by applying the fraction $\frac{1}{4}$ to Table V and $\frac{3}{4}$ to Table VI to derive Table VII. Let x_{ij} represent an element of the solution in Table V—the amount to be shipped from O_i to D_j. Let x'' represent the amount to be shipped from O_i to D_j in Table VI. Then the formula

$$\tfrac{1}{4}x'_{ij} + \tfrac{3}{4}x''_{ij} = x'''_{ij}$$

[9] Actually an infinite number.

[10] Alternatively any three non-negative fractions whose sum is one may be applied to all three of the optimum tables.

yields the amount to be entered into the equivalent cell of Table VII. For example, reference to Table V shows $x'_{11} = 2$; reference to Table VI shows $x''_{11} = 1$. Therefore,

$$\tfrac{1}{4}x'_{11} + \tfrac{3}{4}x''_{11} = \tfrac{1}{4}\cdot 2 + \tfrac{3}{4}\cdot 1 = \tfrac{5}{4} = x'''_{11}\,,$$

the amount entered at O_1 - D_1 of Table VII. Similarly, $x'_{21} = 0$ (Table V) and $x''_{21} = 1$ (Table VI) so

$$\tfrac{1}{4}x'_{21} + \tfrac{3}{4}x''_{21} = \tfrac{1}{4}\cdot O + \tfrac{3}{4}\cdot 1 = \tfrac{3}{4} = x'''_{21}\,,$$

in Table VII. Finally $x'_{31} = 0$ and $x''_{31} = 0$, so that $x'''_{31} = 0$, as indicated by the blank at O_3-D_1 in Table VII. All other values are derived by repeated application of the formula.

The program cost of Table VII is also \$14. Many more such best programs could be derived by choosing different pairs of fractions such as (1/2, 1/2), (1/3, 2/3), whose sum is unity, and applying them to any two of the three basic programs of Tables IV, V, and VI. In this manner all other such programs can be obtained. The reverse is not true. If one of the three basic tables were missing it would not be possible to derive it (or those derived from it) by reference to the others. This supplies a guide to the reason for distinguishing between *basic* and *derived* programs. The basic programs provide a means of obtaining *all* other best programs. The derived programs do not.

13. Degenerate Solutions

Each of the basic solutions has exactly seven circled items. This is a property of any basic solution. The number of circled values will be one less than the number of rows *plus* the number of columns. Moreover, the circled values will be so located in the table that evaluation of every one of the remaining (blank) cells in the table is possible by the computing method which was previously indicated. That is, it must be possible to begin on a "stepping stone" in the same row as the blank cell being evaluated; move out of that row in a straight line on the first move to another stepping stone; and with alternating row and column moves (in the manner of a rook in chess), move from one stepping stone to another until the end of the trail is reached on a stone in the same column as the cell being evaluated. It is important to check, after the circled values have been entered, to see that these conditions are fulfilled—i.e., that it is possible to evaluate every vacant cell in this manner. Moreover, no circled value must have the number zero inside it. When these conditions are not fulfilled, the solution is said to be degenerate.

14. Procedure for Resolving Degeneracy

Fortunately a simple instruction is available for resolving the difficulty should these conditions not be fulfilled at any stage in the computations. A circle (or even several circles) may be arbitrarily entered in the table at the needed point (or points). Inside this circle is entered a small value $\epsilon > 0$. No specific number need be entered. The symbol, ϵ, will suffice. It is merely assumed that the value

TABLE IIA

A Degenerate Solution

Origins	Destinations					
	D_1	D_2	D_3	D_4	D_5	Totals
O_1	−2	1	4	3	ε	5
O_2	−3	1	−1	4	−5	5
O_3	2	3	1	2	4	6
Totals.......	2	2	4	4	4	16

assigned to ϵ is very small relative to the other numbers in the program. It is as though a very small stone were added at the juncture needed to allow the explorer to make his complete tour (and evaluation) of the vacant cells.

In the evaluation process ϵ is treated precisely as though it were a small positive number. Whenever this stone is used as an evaluation path it is multiplied by its cell-mate, c_{ij}, from Table IA and entered (with the appropriate plus or minus sign) in the evaluation sum.

It may be the case that the ϵ-stone entered with a plus sign in the path used to evaluate the most negative sign. Since ϵ is, by definition, the smallest number in the table, it is the one which is moved into the most negative cell as the first step in forming a new table.[11] The remaining circles are, as before, entered in their previous position and then filled out by reference to the rims. In the process of determining these numerical values, ϵ may be ignored. It is treated as insignificant in terms of its effects on the program.

15. Illustration of Rule for Degeneracy

The process may be illustrated by arriving at an alternative first solution to that provided in Table II. If a rule other than the northwest corner rule is used, a program such as the one displayed in Table IIA might be obtained as a first trial solution.

Ignoring ϵ it may be seen that there are only six stones here which provide a solution. Since seven are required the solution is degenerate. To repair the situation an ϵ-stone is entered in the northeast corner, as indicated. It is now possible, by applying the previously stipulated rules, to evaluate each of the vacant cells.

These evaluations are entered as the non-circled items in Table IIA. Negative evaluations appear and so the possibility of program improvement is indicated.[12]

[11] Where more than one ϵ-stone is used the relative smallness of the ϵ's may be determined according to the following conventions: (1) Any ϵ toward the top of the table is smaller than all ϵ's below it; (2) if two (or more) ϵ's occupy the same row the one furthest to the left-hand side of the table is smallest. All questions of choice between smallest stones is, in this manner, unambiguously resolved.

[12] The cost of this program is $29 + 3\epsilon$ dollars, as can be determined by reference to the unit cost Table IA.

The most negative value $-\underline{5}$ was reached by the following steps O_2-D_2, O_1-D_2, O_1-D_5. Hence ϵ entered with a plus sign. As the smallest value in the evaluation path it is moved from cell O_1-D_5 to cell O_2-D_5 as the first step in deriving a new solution.[13] The other circles are entered in blank as before and then filled relative to the rims.

The entrance of an ϵ-stone in no way alters the previous computing instructions. By repeated application of these rules the previously obtained best programs (Tables IV, V, VI, VII) will be achieved. The ϵ will disappear en route. Even if this had not happened (as will be true in some cases at a stage when no negative evaluations are obtained), no difficulty is experienced. The ϵ is merely ignored in delineating the optimum program.[14]

16. Some Limitations of This Method

The discussion here has been centered on a simple problem of transporting goods from a set of origins to a set of destinations. The method of calculation is designed to secure a best program—as measured by least cost—which satisfies the stipulations (along the rims). As contrasted with other methods of calculation it is perhaps more easily comprehended.[15] For other than transportation-type problems a more general approach, known as the simplex method,[16] is available. The simplex technique is more complicated than this transportation method, but it will handle any kind of linear programming problem.[17]

17. Opportunity Cost Evaluations and Alternative Uses

As noted at the outset, linear programming is applicable to a wide variety of business problems, particularly those involving volume-mix considerations. Formally, linear programming provides a means of obtaining best possible programs which satisfy stipulated restrictions. Such restrictions may arise from technological conditions, and market prospects. Technological conditions include such things as worker and machine productivities and capacities arising from the type of plant, chemical process, method and lay-out used.[18]

Policy stipulations may involve program-balance considerations setting maximal and minimal levels to production of certain products or groups of products; restrictions on permissible fluctuations on levels of production with variations (e.g., seasonal) in product demand; maximum and minimum levels of inventory

[13] Since ϵ is considered insignificant relative to the rims, the other circles will retain not only their position but also their values in the new program. The cost of the new program will therefore be $29 - \epsilon$ dollars—a reduction of 4ϵ dollars from the previous level. There is no need to be discouraged by such minute reductions in program cost. No infinite regress is involved. An optimum will always be achieved in a finite number of calculations.

[14] E.g., had this happened in the present case, six rather than seven shipments from origins to destinations would have been indicated.

[15] See supra footnote 1. An exposition of Dantzig's method may also be found in A. Henderson and R. Schlaifer [3].

[16] Devised by G. B. Dantzig. See [4].

[17] In principle this transportation method preserves all of the essential features of the simplex technique. See Appendix.

[18] Many other restrictions such as storage limitations may be included under technology.

TABLE IC

Unit Shipment Costs

Origins	Destinations				
	D_1	D_2	D_3	D_4	D_5
O_1	−2	−1	−2	−3	−3
O_2	−2	−2	−2	−1	$-M$
O_3	−3	−3	−2	−1	−2

accumulations; etc. Market conditions refer to price, quantity, discount considerations, etc.

The methods of linear programming are designed to take into account all possible opportunities which are available within the limits of these restrictions and to determine which particular combinations offer the best, second best, etc., prospects. The methods may also be reversed to assess the desirability of altering the restrictions themselves.[19] Hence, linear programming offers a powerful and flexible instrument of analysis.

The transportation model can be extended to apply to problems where maximal or minimal amounts are available or required at various origins and destinations. By this means storage as well as shipment features may be introduced into the model. Transshipments between origins or between destinations may also be brought under the purview of modified versions of the model. Nor need the model be restricted to transportation problems in the literal sense. The authors have adapted it to problems such as scheduling orders through machines with the objective of minimizing set-up times. Some insight into the possibility of effecting such adaptations is obtained by noting that "payoff" or "cost" matrices (such at Table IA) are not restricted to dollar magnitudes. The cells may contain estimates or assessments of such things as set-up times, ratings of personnel in terms of relative effectiveness on different jobs, etc.

18. An Illustration

A simple modification of Table IA may be introduced in order to provide insight into possible use of linear programming for evaluating facilities.[20] Suppose, for example, that route O_2-D_5, the "subsidy route," is not practical to use—e.g., the subsidy is being offered to encourage a carrier to construct the necessary facilities. A question then arises as to the dollar amount which it is worthwhile for the carrier to invest.[21] It might seem that only \$4 is involved since, in

[19] See [2]. A discussion of how these methods may be used to guide data collection in an economical fashion may be found in [1].

[20] This modification is important in adapting the model to machine scheduling problems where consideration must be given to the fact that some machines may not be suitable for certain kinds of work.

[21] The problem is here stated in static terms (for a single time period) in order to avoid introducing dynamic models.

the optimal program, only 4 units are shipped from O_2 to D_5. This, however, overlooks the interdependence features of this kind of problem. Unavailability of route O_2-D_5 will require distribution of these 4 units over the network so that "bumping" will occur in many parts of the program.

To make O_2-D_5 "impractical" Table IA is rewritten precisely as before except that in place of the \$1 subsidy a large value \$-M is introduced in this cell. This modification is presented in Table IC. No particular numerical value is assigned to M. Just as ϵ was assumed to be insignificantly small, so M is assumed to be dominantly large. Hence, no program can be best which utilizes the cell O_2-D_5. An improvement can always be made by transferring goods scheduled through O_2-D_5 to any other route. A cost of \$$M$ per unit shipped is, therefore, from the viewpoint of best possible programming, equivalent to regarding this route as impractical.

Table IVA-1 presents the best program previously obtained in Table IV. Although the programs are identical it can be seen that the evaluations obtained

TABLE IVA

Program for Route O_2 - D_5 Unavailable

Origins	Destinations					
	D_1	D_2	D_3	D_4	D_5	Totals
			—1—			
O_1	2	2	1	2	3-M	5
O_2	0	1	1	0	4	5
O_3	1	2	2	4	2-M	6
Totals.......	2	2	4	4	4	16
			—2—			
O_1	2	2	1	4-M	3-M	5
O_2	0	1	3	2-M	2	5
O_3	M-1	M	M-2	4	2	6
Totals.......	2	2	4	4	4	16
			—3—			
O_1	2	2	1	2	1	5
O_2	0	1	3	2	M-2	5
O_3	1	2	0	2	4	6
Totals.......	2	2	4	4	4	16

from Table IC indicate that this program is no longer best. Cells O_1-D_5 and O_3-D_5 both contain (with negative sign) the dominatingly large cost, $\$M$.

A further calculation is indicated. The value ② is shifted from cell O_3-D_3 to cell O_3-D_5 where the most negative value $\underline{2\text{-}M}$ is located. Applying the rules previously learned the program indicated in Table IVA-2 is obtained. Again negative values are obtained. The value ② in O_2-D_5 is shifted to the most negative cell and the program of Table IVA-3 is obtained. All values are positive with the impractical route O_2-D_5 containing the dominantly large *positive* value, M-2. This program, which is identical with the one set forth in Table II, is least costly under these conditions.

The cost of the best program with route O_2-D_5 unavailable is \$26, an increase of \$12 over the cost of the program set forth in Table IV. Hence, it would be erroneous to evaluate the worth of this facility at the \$4 which can optimally be earned by shipments over this route when it is available. Such an error might arise from failing to note the interdependencies and resulting shifts necessary to care for other supplies and demands. On a strict opportunity cost basis O_2-D_5, under the projected subsidy, would be worth \$12.[22] The carrier will find it advantageous to pay up to this amount for this facility. All the available alternatives have been considered, as required by the doctrine of opportunity (economic) cost. The basis for determining such costs is provided by a comparison between *best* possible programs with facility O_2-D_5 present and absent.

19. Conclusion

By devising and working through a few simple examples the reader will quickly develop considerable facility with the methods outlined in this paper. He will also be able to devise numerous short cuts. For example, it will quickly become obvious to him that it is not necessary to evaluate all the cells in each table. As soon as a negative cell appears the calculations may be stopped and that cell treated as "the most negative." In applied work where some experience has been gained it is not necessary to begin with the northwest corner rule. Intuition or past programs may be used to provide an initial solution which is much closer to the best. Intuition may also be applied at any stage should better possibilities become apparent. In this manner the number of tables to be calculated may be reduced. A check on these intuitive judgments and assurance of optimality may be attained by applying the rules set forth in this paper.

The person interested in more general cases than transportation-type problems may refer to the citations in the bibliography. The reader should be warned that the methods outlined in this paper are not applicable to all types of linear programming problems. Other types of problems may require recourse to other methods such as the general simplex of G. B. Dantzig [4]. Although the simplex procedure may appear somewhat more complicated than the transportation method outlined here, it, too, can be translated into simple operating rules.[23]

[22] I.e., \$12 per scheduling interval. A decision to construct this facility should, of course, consider the appropriately discounted capital cost over the facility's useful life.

[23] See [2].

One, but only one, class of linear programming problems has been covered in this exposition. As can be seen, the rules for effecting the applications of these linear programming methods are quite simple. An attempt to explain why these results work are set forth in the Appendix to this paper. Reference to this Appendix will show that a fairly complex mathematical analysis underlies (and justifies) these simple rules. Nevertheless, as is often the case, such analyses can, with some effort, be translated into operating rules which can be understood by persons with little or no mathematical training. Such efforts at translation are essential if the full fruits of interaction between theory and application are to be obtained in developing the management sciences.

APPENDIX

The preceding portion of this paper constitutes an attempt to explain, in non-technical terms, one of the new management tools made available by recent scientific research. Justification for these procedures must rest on technical mathematical considerations. The following analysis is, therefore, presented for the benefit of experts in linear programming who may be interested in pursuing the underlying analysis.

The Simplex Method

To establish the relation between the "stepping-stone" method described in this chapter and the simplex procedure, it will be well to begin by recalling the major features of the latter. The general problem of linear programming may be stated as follows: To maximize, or minimize

$$z = \sum_{j=1}^{n} x_j c_j$$

subject to

$$\sum_{j=1}^{n} x_j P_j = P_0 \tag{1}$$

$$x_j \geq 0$$

where the P_j are m-rowed "structural vectors" and P_0 is an m-rowed "stipulations vector."

The simplex method of solution to this problem proceeds as follows:

1. Array the structural and stipulations vectors in the form of a "tableau," adding slack or artificial vectors if necessary to convert the problem into the above form and such that an initial solution is possible in terms of m unit vectors (slack or artificial)—e.g.,

$$P_0 = \sum_{i=1}^{m} x_i P_i , \tag{1a}$$

where the indexes $i = 1, \cdots, m$ have been chosen to designate this selected set of unit vectors.

2. The elements in the j^{th} column in the above tableau are then the coefficients y_{ij} of the expression of P_j in terms of the P_i, e.g.,

$$P_j = \sum_{i=1}^{m} y_{ij} P_i ; \qquad j = 1, 2, \cdots, n. \tag{2}$$

3. An additional row is added. The element in j^{th} column therein is $z_j - c_j$ where

$$z_j = \sum_{i=1}^{m} y_{ij}\, c_i \tag{3}$$

4. Assuming the problem to be one of maximization, the "most negative" $z_j - c_j$ is employed as a guide. Let this element be located in the k^{th} column, with the designation $z_k - c_k$.

a) If this element is positive or zero, an optimum has been achieved.

b) If it is negative, one proceeds further as specified in subsequent rules.

5. Determine the ratios x_i/y_{ik}, for those $y_{ik} > 0$. Let r be the value of i for which

$$\theta = \min x_i/y_{ik} \tag{4}$$

is attained. Then a new solution and new tableau will be determined in which P_r will be replaced by P_k. The new tableau is determined by means of the following formulae:

$$\begin{aligned} x_k' &= \theta \\ x_i' &= x_i - \theta y_{ik}, & i \neq r \\ y_{rj}' &= y_{rj}/y_{rk} \\ y_{ij}' &= y_{ij} - y_{rj}' y_{ik}, & i \neq r \\ z_j' - c_j' &= z_j - c_j - y_{rj}'(z_k - c_k) \end{aligned} \tag{5}$$

6. The new values $z_j' - c_j'$ are inspected. If no negative values are obtained, the calculations are complete and an optimum has been achieved. If negative $z_j' - c_j'$ appear, rule 5 is repeated and successive tableaus are obtained until the criterion of optimization is satisfied. The x' values appearing under P_0, the stipulations vector, provide the solution in the final tableau.

Transportation Model

Consider the problem portrayed in Table IB. The requirements at the three origins and five designations may be stated in the form:

$$\begin{array}{lllllllllllllllll}
x_1 + x_2 + x_3 + x_4 + x_5 & & & & & = 5 \\
& x_6 + x_7 + x_8 + x_9 + x_{10} & & & & = 5 \\
& & x_{11} + x_{12} + x_{13} + x_{14} + x_{15} & & & = 6 \\
x_1 & + x_6 & + x_{11} & & & = 2 \\
x_2 & + x_7 & + x_{12} & & & = 2 \\
x_3 & + x_8 & + x_{13} & & & = 4 \\
x_4 & + x_9 & + x_{14} & & & = 4 \\
x_5 & + x_{10} & + x_{15} & & & = 4
\end{array} \tag{6}$$

The first three equations refer to shipments from origins and the last five refer to receipts at destinations.

Transformation into Double Script Notation

This statement of the problem may be reformulated into the double script notation as follows: Let x_{ij} denote the amount shipped from the i^{th} origin, O_i, to the j^{th} destination, D_j, and let c_{ij} denote the associated unit cost of shipment. In terms of the previous notation, then

$$\begin{aligned} x_{ij} &= x_{(i-1)N+j} \\ c_{ij} &= c_{(i-1)N+j} \end{aligned} \tag{7a}$$

where N is the number of destinations. A corresponding double-script notation may also be introduced for the vector which is the coefficient of x_j in $\sum x_j P_j = P_0$, *viz.*,

$$P_{ij} = P_{(i-1)N+j}\,. \tag{7b}$$

In the double script notation, the problem is to minimize

$$\sum_i \sum_j c_{ij} x_{ij}\,,$$

or what is the same thing, to maximize

$$\sum_i \sum_j (-c_{ij}) x_{ij} \tag{8b}$$

subject to

$$\begin{aligned} \sum_i \sum_j x_{ij} P_{ij} &= P_0 \\ x_{ij} &\geq 0. \end{aligned} \tag{8c}$$

Restatement with Unit Vectors

These rules make it possible to transform conveniently from single to double script notation, and vice versa. By reference to the matrix of equations (6) it may be seen that any column vector may be expressed as

$$P_l = P_{(i-1)N+j} = U_i + V_j \tag{9a}$$

where U_i is the eight-rowed unit vector with unity in row i, $i = 1, 2, \ldots, M$, —(M = number of origins)—corresponding to equation i, which relates to the total amount shipped from O_i; similarly, V_j is the unit vector corresponding to the equation which relates to the total amount received at D_j. Thus,

$$P_l = P_{ij} = P_{(i-1)N+j} = U_i + V_j\,. \tag{9b}$$

Coefficient Values

Because of the special property displayed in (9a) and (9b), if

$$P_l = \sum_{r=1}^{m} y_{rl} P_r\,,$$

where $P_1, \cdots, P_m$ form a basis,[24] the y_{rl} can only be $+1, -1, 0$.

The necessity of this may be illustrated by introducing

$$w_{ij}(s\, t) \equiv y_{rl}$$

where

$$r = (i - 1)N + j, \qquad 0 < j \leq N$$

$$l = (s - 1)N + t, \qquad 0 < t \leq N.$$

Consider, for example, the vector P_{14}, which is not in the solution set of Table II, as evidenced by the fact that no "stepping stone" appears in this cell. Its statement in terms of the basis set displayed in the table becomes:

$$P_{14} = w_{11}(14)P_{11} + w_{12}(14)P_{12} + w_{13}(14)P_{12} + w_{23}(14)P_{23} + w_{24}(14)P_{24} + w_{34}(14)P_{34} + w_{35}(14)P_{35}$$

or,

$$U_1 + V_4 - y_{14}(U_1 + V_1) + y_{24}(U_1 + V_2) + y_{34}(U_1 + V_3) + y_{84}(U_2 + V_3) + y_{94}(U_2 + V_4) + y_{14,4}(U_3 + V_4) + y_{15,4}(U_3 + V_5).$$

Reference to the definitions[25] of U_i and V_j reveals that U_1 and V_4 can only be obtained from these same unit vectors appearing on the right-hand side.[26] The solution is given by $y_{34} = 1$, $y_{84} = -1$, $y_{94} = 1$ and all other $y_{ij} = 0$. That is, the coefficients are all $+1, -1$, or 0. It will be noted, moreover, that the non-zero coefficients are associated with P_{13}, P_{23}, and P_{24}—the stepping stones used in the path about $P_{14} = U_1 + V_4$, or P_4 in single index—i.e., general simplex notation.

The stepping-stone procedure starts with a "stepping stone" or "basis" vector in the same row, s, which yields U_s with coefficient unity. It also produces an extraneous V_{j_1}. To remove this one hops to another stepping stone (vector) in the same column and attaches the coefficient (-1). This eliminates V_{j_1} but introduces another extraneous vector U_{s_1} with coefficient (-1). Proceeding in this manner with alternate column and row moves the successive extraneous vectors are eliminated yielding, finally, $U_s + V_t$ upon coming to rest in the same column as the vacant cell which is being evaluated. (The evaluation is, of course, obtained by inserting the appropriate unit costs in place of the P_{ij}.)

The indexes i and j in the expression

$$P_{st} = \sum_i \sum_j w_{ij}(st)P_{ij}$$

range over only those values which give the "basis" P_{ij}'s. Hence, the $w_{ij}(s\, t)$'s are unique. Since the "stepping stone" method assigns possible values to these

[24] Using the single script notation—it turns out that $m = N + M - 1$.

[25] See (9a).

[26] Degeneracy is revealed when it is not possible to insert the necessary U_i or V_j in the right-hand expression—hense, making it impossible to obtain an expression for P_{ij}.

w's they must be the correct ones. In particular, the stepping-stone vectors not in the evaluating path of P_{st} have $w_{ij}(s\,t) = 0$.

In closing this discussion it might be noted that there is no necessity for division in finding θ—see rule (5) of the simplex method—since the positive $w_{ij}(s\,t)$ are all $+1$.

REFERENCES

1. Charnes, A., Cooper, W. W. and Farr, D., "Linear Programming and Profit Preference Scheduling for a Manufacturing Firm," *Journal of the Operations Research Society of America*, Vol. 1, No. 3, May, 1953.
2. Charnes, A., Cooper, W. W. and Henderson, A., *An Introduction to Linear Programming*, John Wiley & Sons, Inc., New York, 1953.
3. Henderson, A. and Schlaifer, R., "Mathematical Programming, Better Information for Better Decision Making," *Harvard Business Review*, Vol. 32, No. 3, May–June, 1954.
4. Koopmans, T., ed. *Activity Analysis of Production and Allocation*, Cowles Commission for Research in Economics No. 13, John Wiley & Sons, Inc., New York, 1951.

THE USE OF MATHEMATICS IN PRODUCTION AND INVENTORY CONTROL

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It is becoming increasingly apparent that some new scientific and mathematical theories are now being developed in this country which, in combination with the remarkable performance of electronic data processing machines, will bring about significant changes in current managerial techniques. The field of production and inventory control is particularly amenable to these new theories, as evidenced by the extensive literature devoted to this subject, and, therefore, it can be expected that a reorientation of production control methods will be in order in various segments of American industry in the not too distant future. How distant in the future is not known, but it is clear that the tempo of these changes will greatly depend on the degree of integration effected between the production men, the management scientist, and the electronic engineer.

A particularly significant difficulty to be overcome is the successful transmission of information between the production men and the scientist developing new theories of production. While it is not to be expected that operating personnel will find it necessary to delve into the fine details of some of these mathematical techniques, it still is mandatory that a clear concept of the nature of these methods be understood. Unfortunately, there has been very little effort expended in "popularizing" these mathematical concepts, and, in fact, there might have been an implication that it is hopeless to transmit to the "layman" the essence of these techniques.

In the course of studying production and inventory control, the author of this paper found it repeatedly necessary to explain his mathematical theory to operating personnel, many of whom have had little or no formal training in mathematics at the college level. This paper, then, is an outgrowth of these presentations and has a twofold purpose: (1) to present a theory of a very small part of the problem of production and inventory control with the objective of acquainting the reader with the nature of the mathematical methodology, (2) to lay emphasis on a didactic presentation to show the principles involved in explaining these mathematical concepts.

The subject matter to be discussed is what is generally called the preparation of parts requirement lists and production explosion charts.

In order to fix ideas, we paraphrase the problem in question as follows:

> The manufacturing planning program begins with breaking down the sales forecast into the requirements for the detailed subassemblies and parts. A production explosion chart is plotted showing, for a manufacturing unit, first, the breakdown into major assemblies and separate parts, and then at each successive step the further breakdown into subassemblies

PANEL 435090012

3	420990309		BUSHING
1	435090012	1	PANEL BLANK
2	435090012	2	ANGLE
1	435090012	7	ANGLE
5	99967C096		RECEPTACLE
10	AN426AD3		RIVET
8	AN426AD4		RIVET

FIGURE 1

THIS SHEET REFERS TO THE ASSEMBLY "PANEL" WHICH IS ASSIGNED THE PART NUMBER 435090012. THIS PANEL IS MADE UP OF SEVEN DIFFERENT ARTICLES WHICH COULD BE SUBASSEMBLIES OR PARTS. THE PANEL REQUIRES THREE BUSHINGS 420990309, AND ONE PANEL BLANK 435090012-1, ETC. EACH ASSEMBLY WILL HAVE A SIMILAR SHEET. THE TYPE OF INFORMATION CONTAINED ON THESE SHEETS WILL FORM THE BASIS OF OUR DISCUSSION.

ASSEMBLY PARTS LIST

and parts. Based on this "explosion system" the requirements of all assemblies and sub assemblies are determined before any parts manufacturing orders are initiated.[1]

Every production man knows that a great deal is implied in the above statement; however, the statement contains little clue as to what these implications are. The fact of the matter is that many pages of verbal discussions of the subject are required to develop these implications and in many instances, from the point of view of operating personnel, there is no complete description available in the literature at all. In contrast, we proceed here to develop a statement in terms of mathematical concepts which not only suggest procedures and methods but in fact specifically contain answers to some of the fundamental issues involved in the above statement.

In order to develop such a theory it is necessary to examine these questions in detail and to, step by step, build a bridge from this above statement to mathematical equations. That such a thing can be done and that it can be useful, most likely is a perplexing thought to the reader.

The Concept of the Assembly Parts List

We begin by considering Figure 1 where a sample assembly parts list is shown —each assembly to be made has a similar sheet. The assembly in question is a "Panel" with Part No. 435090012. This number is shown on the upper right-hand

[1] *Production Handbook*, L. P. Alford and J. R. Bangs, p. 230, "Breaking Down Requirements."

ARTICLE		435090012
3 OF	ARTICLE	420990309
1 OF	ARTICLE	435090012-1
2 OF	ARTICLE	435090012-2
1 OF	ARTICLE	435090012-7
5 OF	ARTICLE	99967C098
10 OF	ARTICLE	AN426AD3
8 OF	ARTICLE	AN426AD4

FIGURE 2

ABBREVIATED ASSEMBLY PARTS LIST;
the names of the various articles are omitted.

corner under the word "Makes Assembly." This assembly is made up of seven different articles[2]—bushing, panel black, etc. The part number of each of these is given on the sheet. Under the heading N. A. QTY. (next assembly quantity) it also is shown how many of these articles are needed. Thus, the bushing with Part No. 420990309 is required in a quantity of three for each of the Panels 435090012.

The Assembly Parts List has some other information which for the purpose of our simplified discussion is disregarded.

In order to build a mathematical model, the information in the Assembly Parts List is to be put in a concise symbolic form. Note first that the information in Figure 1 is redundant; the panel in question has the Part Number 435090012 and, therefore, this assembly could be identified solely by this number.

Figure 2, then, presents an abbreviated parts list; the names of the articles are no longer listed.

In the remainder of the report, for the sake of brevity, we shall not carry these long part numbers, but shall assume that the articles are numbered 1, 2, 3, etc. With these short-cuts, then, a set of assembly parts lists might take the form of Figure 3. This purely hypothetical manufacturing process (which bears no relationship to the one described in Figures 1 and 2) deals with Assemblies 1, 2, 4, 5, 7, 8, and 9. Assembly 1 is made up of one of article 3 and two of article 5. Each column of Figure 3 represents a single assembly parts list similar to Figures 1 or 2.

A further saving of words can be effected by saying A instead of article and

[2] An "article" might be an assembly, subassembly, or part.

ARTICLE	1	2	4	5	7	8	9
	1 OF ARTICLE 3	2 OF ARTICLE 6	2 OF ARTICLE 1	3 OF ARTICLE 3	1 OF ARTICLE 1	1 OF ARTICLE 1	3 OF ARTICLE 6
	2 OF ARTICLE 5	1 OF ARTICLE 7	1 OF ARTICLE 7	1 OF ARTICLE 6	2 OF ARTICLE 5	1 OF ARTICLE 5	1 OF ARTICLE 8
		2 OF ARTICLE 8					

FIGURE 3

SET OF ABBREVIATED ASSEMBLY PARTS LISTS

Each column represents a single assembly parts list. For instance, article 2 is made up of two of article 6, one of article 7, and two of article 8.

A_1	A_2	A_4	A_5	A_7	A_8	A_9
$1A_3$	$2A_6$	$2A_1$	$3A_3$	$1A_1$	$1A_1$	$3A_6$
$2A_5$	$1A_7$	$1A_7$	$1A_6$	$2A_5$	$1A_5$	$1A_8$
	$2A_8$					

FIGURE 4

SET OF ABBREVIATED ASSEMBLY PARTS LISTS

The word Article 1 is replaced by A_1, article 2 by A_2, etc.

saying A_1 for article 1, A_2 for article 2, etc. With this notation then Figure 3 becomes Figure 4.

The information in Figure 4 can be presented in a somewhat different form using rectangular tables. Such a table is shown in Figure 5. The information is the same as in Figure 4; for instance, it can be seen that A_4 is made up of two A_1's and one A_7. The advantage of the new presentation is that it is conceptually more descriptive. One can talk about a column, describing what an assembly is made of, or one can talk about a row, showing what an assembly goes into. For instance, A_1 is made of one A_3 and two A_5's; A_5 goes into A_1 twice, into A_7 twice, into A_8 once, and into A_9 three times.

The same information is finally presented again in Figure 6. What we did is

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
A_1				2			1	1	
A_2									
A_3	1				3				
A_4									
A_5	2						2	1	3
A_6		2			1				
A_7		1		1					
A_8		2							1
A_9									

FIGURE 5

TABLE OF ASSEMBLY PARTS

The information still is the same as on Figure 4 but the presentation is more systematic. Note that for completeness all the articles are listed in the top now. For instance, A_3 requires no other article as A_3 is not an assembly but a part.

very simple; we filled in the empty squares with zeros and omitted the *A*'s. This presentation is very simple because we can say that every article goes into every other article—the numbers on the table show how many times. The zeros mean that a particular article does not "really" go into the other article in the ordinary sense; from our point of view, this distinction need not be made.

For explanation, let us insert an analogy from algebra. We can subtract any two numbers, five less five equals zero. If we did not have zeros we would have to say that subtraction can be carried out only under certain circumstances. We always would have to watch that the formulas have meaning. Therefore, the invention of the zero is an extremely useful thing. It also forms the essential foundation of the concept of Arabic numbers as distinct to Roman numerals where zeros do not exist. Anyone who would have the courage to carry through a division in Roman numerals would appreciate the point.

One further point—we have put zeros into the "diagonal" elements[3] of the table. The question of how many A_2's go into A_2 is not a significant one and we could adapt any convenient system. Later, however, it becomes clear that using zeros makes the mathematics simple. In Figure 6, it can be seen that a row of zeros (e.g., the fourth row) indicates a top assembly; A_4 does not go into anything. A column of zeros indicates a detail part; thus, Article 6 does not require anything since it is not an assembly, but a part.

[3] The diagonal elements are formed by the first number of the first row, the second number of the second row, the third number of the third row, etc.

	1	2	3	4	5	6	7	8	9
1	0	0	0	2	0	0	1	1	0
2	0	0	0	0	0	0	0	0	0
3	1	0	0	0	3	0	0	0	0
4	0	0	0	0	0	0	0	0	0
5	2	0	0	0	0	0	2	1	0
6	0	2	0	0	1	0	0	0	3
7	0	1	0	1	0	0	0	0	0
8	0	2	0	0	0	0	0	0	1
9	0	0	0	0	0	0	0	0	0

FIGURE 6

NEXT ASSEMBLY QUANTITY TABLE

This is a concise mathematical representation of the information contained in the Assembly Parts Lists and this Table will form one of the building blocks of the mathematical theory.

Let us stop for a moment now as we have reached in fact our first objective; the information contained in the Assembly Parts Lists has been put into an appropriate form for further mathematical discussion. Instead of Assembly Parts Lists, we will talk in terms of the Next Assembly Quantity Table as represented in Figure 6.

We proceed now to the determination of the parts requirements, that is, we develop a formula which tells us how many of each assembly and each part is required to meet any sales forecast.

The Total Requirement Factor Table

Before we discuss the determination of parts requirement we introduce some visual aids to clarify our concepts. Consider for this purpose Figure 7. It can be seen that A_1 is made up of one A_3 and two A_5's. A_2 is made up of two A_6's, one A_7, and two A_8's, etc. All this information is contained in the Next Assembly Table in a numerical fashion. Suppose now, we want to know all the articles that are required for each A_2.[4] The resultant diagram is shown in Figure 8. This is not very convenient and so we present the same information in Figure 9 in a different form. Note that each article is shown only once. A_5 goes into A_7 twice so we have two arrows on the line going from A_5 to A_7. The insert shows that, say, A_5 goes into A_1, A_7, and A_8, twice, once, and twice, respectively. Figure 9 is a pictorial representation of certain columns of the Next Assembly Quantity Table; in order to fix our ideas it will be called the Gozinto Graph for A_2. Similar

[4] We mean by the statement, "articles required for each A_2," all the articles that go directly into A_2, *and* all the articles that go into these, *and so on*.

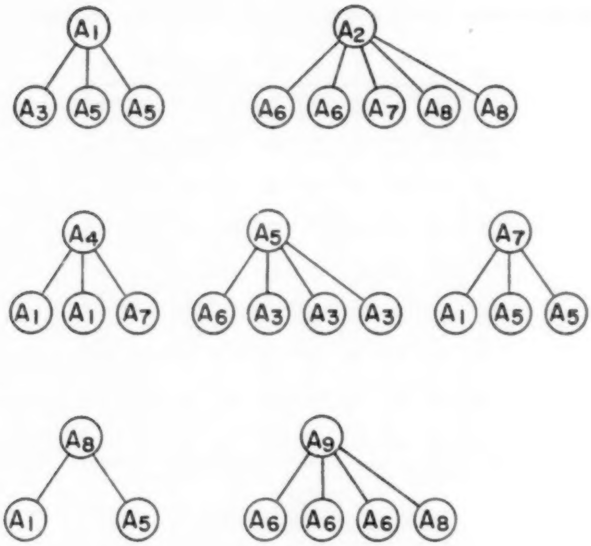

FIGURE 7

PICTORIAL REPRESENTATION OF NEXT ASSEMBLY QUANTITIES

It can be SEEN that A_1 is made up of one A_3 and two A_5's; A_2 is made up of two A_6's, one A_7, and two A_8's. It cannot be seen how many assemblies and parts are required in total, say, making up an A_2, when it is recognized that A_7 and A_8 are assemblies.

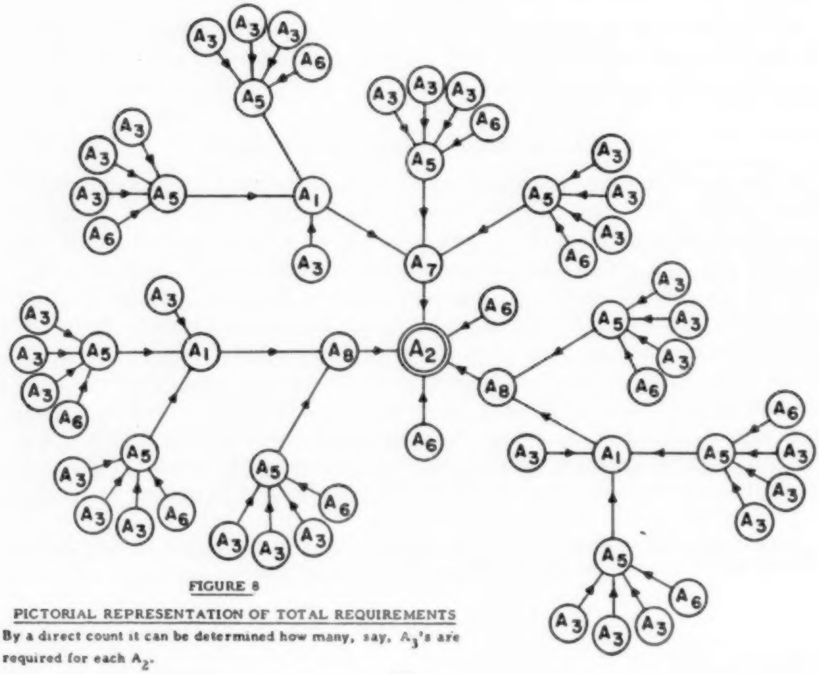

FIGURE 8

PICTORIAL REPRESENTATION OF TOTAL REQUIREMENTS

By a direct count it can be determined how many, say, A_3's are required for each A_2.

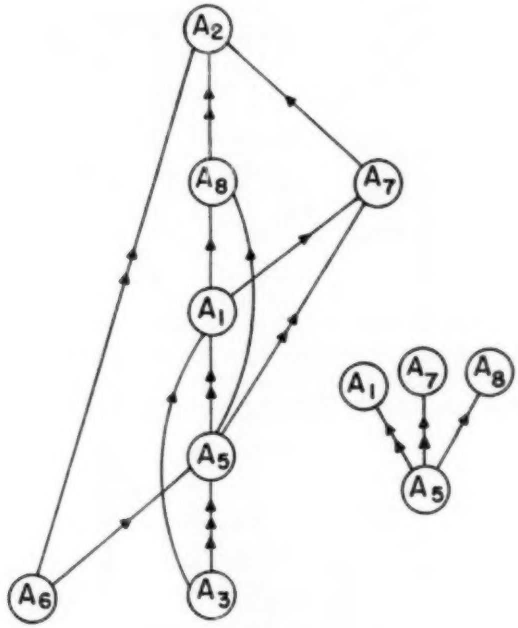

FIGURE 9

THE GOZINTO GRAPH FOR A_2

is a pictorial representation of the requirements for A_2. Each article is shown only once. The next assembly quantities are shown by the multiplicity of the arrows. Total requirements cannot be observed directly but can be deduced.

pictorial representations can be prepared for A_4 and A_9, and a composite Gozinto Graph for all our articles is shown in Figure 10. Figure 9 is much more simple than Figure 8 but contains the same information. However, from one point of view it is not quite so convenient. How many A_6's are required (in total) for each A_2? In Figure 8 one can answer this question by direct count of the A_6's. In Figure 9 some mental effort is required to answer the same question. However, this is the type of thinking that leads to the result we want as when dealing with thousands of assemblies we are not able to draw these various pictures. Suppose we wanted to figure out how many A_5's are required for each A_2. A_5 goes directly into A_1, A_7, and A_8 (see Figure 9), so we can make the statement that the

Total number of A_5's required for each A_2 =
[Number of A_5's going directly into each A_1]
$\cdot$[Total number of A_1's required for each A_2]
+ [Number of A_5's going directly into each A_7]
$\cdot$[Total number of A_7's required for each A_2]
+ [Number of A_5's going directly into each A_8]
$\cdot$[Total number of A_8's required for each A_2]

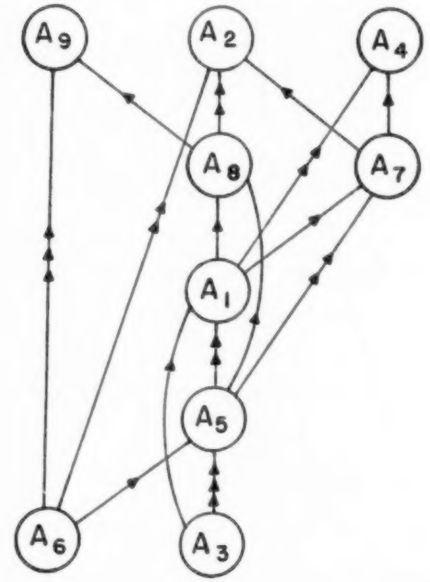

FIGURE 10

THE COMPOSITE GOZINTO GRAPH

is a pictorial representation of the parts requirements. The next assembly quantities can be observed directly by counting the arrows on each connecting line. Total requirements cannot be observed directly but can be deduced.

Note the important difference between these two statements—"Number of A_5's going directly into each A_8," and the statement "Total number of A_5's required for each A_2." The answer to the first statement is the Quantity 1, while the answer to the second one is the Quantity 3. In our Next Assembly Quantity Table we have the number of assemblies going directly into each other assembly listed, but we do not have the total number of assemblies required for each other assembly.

Let us contemplate the above statement. It gives a relationship between total number of quantities required and next assembly quantities. It does not tell us how to compute the total number of quantities required from the next assembly quantities, as the various total numbers required and the next assembly quantities appear at both sides of the equation.

The same statement also implies some sort of a rule as we could also figure a relationship for how many A_5's we need, say, for each A_8. Very likely, this rule could be described in words; however, if one attempted to work out this rule, it would get lengthy and confusing. Clearly, what we need is a concise notation to describe the idea represented. And this is the point where mathematics comes in handy. Consider the diagram below:

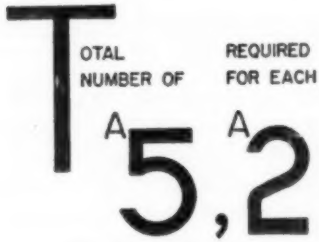

DIAGRAM I

We have magnified the statement "Total number of A_5's required for each A_2." The way the picture was prepared suggests that instead of the long sentence, we could simply say $T_{5,2}$. Quite similarly the diagram below suggests that instead of saying "Number of A_5's going directly into each A_1," we should say $N_{5,1}$.

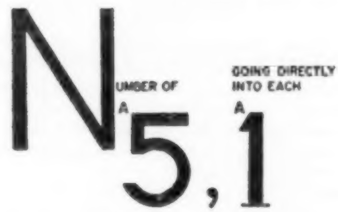

DIAGRAM II

At this point bear in mind that the notation means nothing more or less than what we said. It is a concise statement of things we have known. However, we can now replace the statement in question with an equation:

$$T_{5,2} = N_{5,1} \cdot T_{1,2} + N_{5,7} \cdot T_{7,2} + N_{5,8} \cdot T_{8,2}$$

We have accomplished, then, our first objective—our specific statement is represented in a shorthand form.

Suppose for the moment that we know the answer we are attempting to find—that is, we have computed all the total requirements, which are all the various "T's." They can be put in a table as shown in Figure 11. This table which we call the Total Requirement Factor Table, or briefly, the T Table, is very similar to the Next Assembly Quantity Table. Just as the latter one shows the "N's," the new table shows the capital "T's." One can see, for instance, that each A_7 requires one A_1, thirteen A_3's, nine A_5's, four A_6's, and one A_7.

Let us recognize that once the Total Requirement Factor Table is determined, our problem of answering the question of "how many" becomes very simple. Therefore, let us turn our attention to the general formulation of our equation, which formulation will lead directly to the determination of the Total Requirement Factor Table.

	1	2	3	4	5	6	7	8	9
1	1	3	0	3	0	0	1	1	1
2	0	1	0	0	0	0	0	0	0
3	7	33	1	27	3	0	13	10	10
4	0	0	0	1	0	0	0	0	0
5	2	10	0	8	1	0	4	3	3
6	2	12	0	8	1	1	4	3	6
7	0	1	0	1	0	0	1	0	0
8	0	2	0	0	0	0	0	1	1
9	0	0	0	0	0	0	0	0	1

FIGURE 11

TOTAL REQUIREMENT FACTOR TABLE

Observe say the second column relating to A_2. The third element from the top in this column relates to A_3 and displays the number 33. This means that thirty-three A_3's are required (in total) for each A_2. This can be confirmed by a direct count from Figure 8. (We have not explained yet how this above Table was computed.)

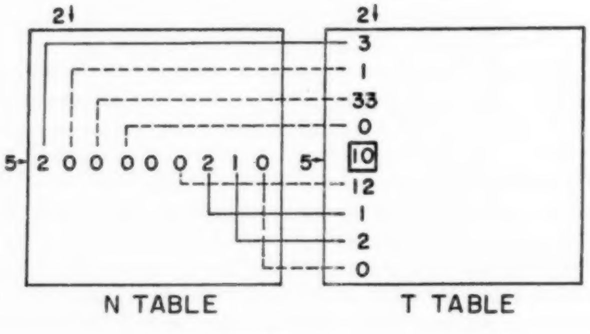

FIGURE 12

SCHEMATIC REPRESENTATION OF THE EQUATION $T_{5,2} = \sum_k N_{5,2} T_{k,2}$

The second number of the fifth row of the T Table equals the "scalar multiple" of the fifth row of the N Table and second column of the T Table:

$10 = 2 \times 3 + 0 \times 1 + 0 \times 33 + 0 \times 0 + 0 \times 10 + 0 \times 12 + 2 \times 1 + 1 \times 2 + 0 \times 0$

Determination of the "T" Table

Assuming for the moment that we have already computed all the T's, let us focus our attention, say, on the second column of the T Table. Figure 12 shows the fifth row of the N Table and the second column of the T Table. Consider $T_{5,2}$, that is, the second element of the fifth row. We can say, as it is shown by

our equation above, that $T_{5,2}$ can be obtained by taking the left-hand side number from the N Table and multiplying it by the top number on the T Table; to this number we have to add the seventh number from the N Table multiplied by the seventh number from the top of the T Table; finally, we have to add the eighth number from the T Table multiplied by the eighth number of the T Table. This rule can be stated in a more simple form: Multiply the first number in the N Table with the first number in the T Table, multiply the second number with the second number, the third with the third, and so on, keeping in mind that we are combining a row with a column. This last rule is, of course, the same as the first, but it takes advantage of the zero's we have in the table.

Incidentally, to make this a good rule, we need both $N_{5,5}$ and $T_{2,2}$. The former was defined as zero on page 74, and now we have to define $T_{2,2}$. We are at liberty to use any number, as the question of how many A_2's are required for each A_2 is not a significant one. However, it turns out that in order to make our rules uniform we have to assign to the diagonals of the T Table the value "1." It was precisely the same reason that we assigned the zeros to the diagonals of the N Table on page 75.

Mathematicians have developed a shorthand notation for sums of the kind we are discussing here. They simply write

$$T_{5,2} = \sum_k N_{5,k} T_{k,2}$$

using for summation the Greek capital letter Σ. This equation means exactly the same thing as the former one. The letter "k" indicates that the product should be computed for all values of "k."

It is quite clear that the equation that we have here works not only for $T_{5,2}$, but for any element on the T Table. This can be written in mathematical form as

$$\boxed{T_{i,j} = \sum_k N_{i,k} \cdot T_{k,j} \quad i \neq j} \qquad (1)$$

The letters i and j simply mean that i can be any number and that j can be any number, though it is postulated that i must be different from j, as in that case $T_{i,i}$ takes the value "1."

Let us try this formula—consider, say, $i = 4$, $j = 2$; the T in question is $T_{4,2}$, the second element in the fourth row of the T Table. According to our rule, we have to combine the fourth row of the N Table (Figure 6) with the second column of the T Table (Figure 11). However, the fourth row of the N Table is a zero. Therefore, we can conclude the $T_{4,2}$ must be zero. This is not surprising as A_4 is a top assembly and A_2 does not require any A_4's.

Quite similarly we deduce from

$$T_{9,2} = \sum_k N_{9,k} \cdot T_{k,2}$$

that $T_{9,2}$ equals zero.

Let us continue now the computation of the rest of the elements in the second column of the T Table. We get

$$\begin{aligned}
T_{8,2} &= \textstyle\sum_k N_{8,k} \cdot T_{k,2} = N_{8,2} \cdot T_{2,2} + N_{8,9} \cdot T_{9,2} \\
&= 2 \times 1 + 1 \times 0 = 2,^{5} \\
T_{7,2} &= \textstyle\sum_k N_{7,k} \cdot T_{k,2} = N_{7,2} \cdot T_{2,2} + N_{7,4} \cdot T_{4,2} \\
&= 1 \times 1 + 1 \times 0 = 1 \\
T_{1,2} &= \textstyle\sum_k N_{1,k} \cdot T_{k,2} = N_{1,4} \cdot T_{4,2} + N_{1,7} \cdot T_{7,2} + N_{1,8} \cdot T_{8,2} \\
&= 2 \times 0 + 1 \times 1 + 1 \times 2 = 3 \\
T_{5,2} &= \textstyle\sum_k N_{5,k} \cdot T_{k,2} = N_{5,1} \cdot T_{1,2} + N_{5,7} \cdot T_{7,2} + N_{5,8} \cdot T_{8,2} \\
&= 2 \times 3 + 2 \times 1 + 1 \times 2 = 10 \\
T_{6,2} &= \textstyle\sum_k N_{6,k} \cdot T_{k,2} = N_{6,2} \cdot T_{2,2} + N_{6,5} \cdot T_{5,2} + N_{6,9} \cdot T_{9,2} \\
&= 2 \times 1 + 1 \times 10 + 3 \times 0 = 12
\end{aligned}$$

and finally

$$\begin{aligned}
T_{3,2} &= \textstyle\sum_k N_{3,k} \cdot T_{k,2} = N_{3,1} \cdot T_{1,2} + N_{3,5} \cdot T_{5,2} \\
&= 1 \times 3 + 3 \times 10 = 33
\end{aligned}$$

We can see, therefore, that Equation 3 indeed allowed us to compute the second column of the T Table. Furthermore, it is clear that Equation 3 also can be used to compute all the other numbers on the T Table; therefore, we can conclude that Equation 3 does contain the necessary instruction for the determination of the T Table.

One more remark before we discuss our result in detail. It is to be pointed out that the computations had to be done in a very particular *sequence* since the T's appear on both sides of Equation 3. However, this need not be a cause for worry as one can try to compute the top element in the column; if this is not possible try the one below, proceed down to the bottom, and then again start on the top. This procedure eventually leads to all the numbers in the column. This method might not be the most efficient but it always works. A more careful study of the problem can lead to a quicker procedure, but we shall not go into the details here.

The Mathematical Form of a Sales Forecast

We have said before that, once the Total Requirement Factor Table is determined, the problem of parts requirements is easy to solve. We propose to establish now the necessary mathematical formulism to determine the parts requirements.

In order to fix ideas, let it be supposed that twenty of A_2, thirty of A_4, eighty of A_9, and fifty of A_5 are specified by the sales forecast, and the problem is to determine how many A_8's are required. Clearly we have

[5] Note that $T_{2,2} = 1$.

Quantity of A_5's required =

(Total number of A_5's required for each A_2)
  $\times$ (Sales forecast of A_2)
+ (Total number of A_5's required for each A_4)
  $\times$ (Sales forecast of A_4)
+ (Total number of A_5's required for each A_9)
  $\times$ (Sales forecast of A_9)
+ (Sales forecast of A_5)

We can put this statement into a mathematical formula. Let $S_1, S_2, S_3, \cdots$ etc., denote the sales forecast for articles $A_1, A_2, A_3 \cdots$ etc, and let the unknown requirements for A_5 be denoted by X_5, then

$$X_5 = T_{5,2} \cdot S_2 + T_{5,4} \cdot S_4 + T_{5,9} \cdot S_9 + S_5 .$$

In our particular numerical case, we get

$$730 = 10 \times 20 + 8 \times 30 + 3 \times 80 + 50.$$

Using again the summation notation, the last equation can be written as

$$X_5 = \sum_k T_{5,k} \cdot S_k$$

where advantage is taken of the fact that $T_{5,5}$ equals 1. Finally, it is clear that a similar equation holds for any article, and so we write

$$\boxed{X_i = \sum_k T_{i,k} \cdot S_k} \qquad (2)$$

In actual practice many of the S's are zero as only some of the articles (say top assemblies and spares) are shippable.

In order to be sure that we understood our formula, let us try the case when $i = 3$. We get

$$X_3 = \sum_k T_{3,k}S_k = T_{3,2} \cdot S_2 + T_{3,4}S_4 + T_{3,5}S_5 + T_{3,9}S_9$$

and

$$2440 = 33 \times 20 + 27 \times 30 + 3 \times 50 + 10 \times 80$$

showing that 2440 A_3's are required.

The Final Mathematical Formulation

We have "solved" the problem of parts requirements in terms of the N and T "tables," our solution being expressed by Equations 1 and 2. As it happens, mathematicians have studied such "tables" in detail—they call them "matrices"—and a "Theory of Matrices" has been developed. In Matrix Algebra, rules for the addition, subtraction, multiplication, and division of matrices are developed.

We cannot go into the details of such a theory, but we will point out here that Equation 1 can be written as

$$[T] = [N][T] + [I]$$

or

$$[T] = \frac{[I]}{[I] - [N]}. \tag{3}$$

where $[I]$ is the so-called "unit matrix." Equation 2 can be written as

$$[X] = [T] \times [S]. \tag{4}$$

Finally, the two equations can be combined into a single equation

$$[X] = \frac{[I]}{[I] - [N]}[S] \tag{5}$$

this being the final mathematical formulation of our problem. In order to bring into focus the mathematical methodology, we make now a complete statement using the language of a mathematician:

Consider the manufacture of articles $A_1, A_2, \cdots$ and denote by $N_{i,j}$ the number of A_i's going directly into A_j. Let $S_1, S_2, \cdots$ denote the sales forecast and let $X_1, X_2, \cdots$ denote the (unknown) parts requirements. Then

$$\boxed{[X] = \frac{[I]}{[I] - [N]}[S]}$$

where $[N]$ is the (square) matrix formed by the $N_{i,j}$'s, $[I]$ is the unit matrix, $[S]$ and $[X]$ are the column matrices formed by the S's and X's.

Concluding Remarks

We have reached the end of our presentation—we have accomplished the transformation of our original verbal statement into a mathematical form. However, let us remind ourselves that we have studied only a very small part of the problem of production and inventory control and that the practical value of the theory lies in its extension to more complicated problems. As an example, we mention that our principal formula can be generalized to include the problems of scheduling. Instead of the sales forecast, $S_1, S_2, \cdots$, one must deal with the sales forecast *functions* $S_1(t), S_2(t), \cdots$, where each of these functions describes the sales forecast for individual planning periods such as days, weeks, months, etc. The requirements $X_1, X_2, \cdots$ are replaced by the requirement functions $X_1(t), X_2(t), \cdots$, designating the requirements for each planning period. The equations relating the $X(t)$'s to the $S(t)$'s become much more complicated as the effect of the various lead times, make spans, inventory policies, etc., all must be

incorporated into the mathematics. A more detailed discussion of these problems lies beyond the scope of this paper and future publications are planned to report on this work. We conclude this paper now by elaborating on some of the advantages offered by the mathematical theory.

I. The mathematical statement deals with clear-cut and precise concepts. For instance, compare the exactness of the ideas involved in matrix multiplication with the vague thoughts represented by the word "explosion." The mathematical theory leads to a comprehension of matters that cannot be obtained by verbal discussions. A more and more complete mathematical statement of managerial problems and the mathematical solution of these problems will lead to an insight never heretofore realized.

II. A further outcome of the foregoing is the possibility of better ways of transmitting information on certain industrial practices. The writer has been constantly impressed by the great barriers that exist between various operating people and departments. The fact of the matter is that many systems and procedures are too complicated to be adequately described in words and, therefore, transmission of their description becomes very difficult. On the other hand, there is the likelihood that a mathematical formulation can be easily explained, as experienced by (in the rare instances of) lucid publications in scientific and engineering fields. The fact that current operating personnel are not trained along mathematical lines should not be considered an insurmountable obstacle.

III. Once the ideas are represented in mathematical form, specific managerial problems may be answered by manipulating the mathematical formulas with the aid of known mathematical techniques. These mathematical techniques need not be practiced by the operating personnel, as mathematicians well trained along these lines could be employed. It is to be emphasized that the mathematics used are not beyond the usual training of a mathematician with a Bachelor's or a Master's degree.

IV. In the introduction we touched upon the role that electronic computing machines will play in industry. However "intelligent" these machines will be, it still will be necessary to prepare the problems to be solved in a language digestible to the computing machine. Many current procedures are not completely formulated and are transmitted by verbal instructions or examples; clearly, this will not be sufficient for the electronic computing machine. Do we face the problem here of training our electronic computing machine experts in all ramifications of managerial tasks? A mathematical formulation give great comfort in solving this problem. For instance, our mathematical formulation of the Parts Requirements problem can be given to a person trained in the use of electronic computing machines and he can solve the problem without going into the mass of details related to actual practices.

V. Finally, let us point out that such mathematical theories will lead to the general understanding of some managerial problems that so far have been treated only in an intuitive or haphazard fashion. Each result obtained in a mathematical theory will form a building block in the structure which eventually will form a discipline that truly could be called by the name "Management Sciences."

SMOOTH PATTERNS OF PRODUCTION*

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1. A number of authors have considered mathematical models for scheduling rates of production to meet a projected series of requirements. In particular, linear models have been used in [1], [2], [4]. This paper considers a simple model of that type, which explicitly allows for inventory costs.

While situations can be found to which the naive model discussed here may usefully be applied, it is undoubtedly not adequate for planning output in most business firms. Nevertheless, the authors feel that the approach is of value for several reasons. In the first place, in many situations the increase in unit cost with level of output may be more directly related to the increase in level than to the absolute level, as assumed in the Modigliani-Hohn model. This could happen, for example, where training of new employes or provision of additional capacity is the main reason for the higher unit cost. Secondly, the model itself is of some interest for its mathematical properties, several of which are suggestive of special attacks on linear programming problems of suitable type.

In section 2 we present the formulation of the linear model for production planning. In subsequent sections, the properties of the models are discussed. The principal results of the paper are Theorem 3, which states a characteristic of the solution to this and related models, and Theorem 4 and its corollary, which gives an explicit and simple solution in the special case where the requirements are monotone increasing.

2. Let $r_1, r_2, \cdots, r_n$ be given positive constants. These are the shipping requirements for the various months. Let $x_1, x_2, \cdots, x_n$ be non-negative variables, representing production in the various months. If we let $R_t = \sum_{i=1}^{t} r_i$, $t = 1, \cdots, n$ and $X_t = \sum_{i=1}^{t} x_i$, $t = 1, \cdots, n$, then it is clear that the stipulation that the shipping requirements be fulfilled may be stated as

$$X_t \geqq R_t, \qquad t = 1, \cdots, n. \tag{1}$$

We treat storage as follows: whatever part of the total production at the end of the t-th month, that has not been shipped is stored. Denote this amount by s_t. Thus

$$X_t - R_t = s_t, \qquad t = 1, \cdots, n. \tag{2}$$

The cost of increasing production from one month to another is described by means of the following notation: if a is a real number, let $a_+ = \frac{1}{2}(a + |a|)$;

* This research was supported by the USAF, through the Office of Scientific Research of the Air Research and Development Command.

i.e., $a_+ = a$ if a is nonnegative, $a_+ = 0$ if a is negative. Then the cost of increasing production from one month to another is

$$(x_t - x_{t-1})_+ , \qquad t = 1, \cdots, n.$$

where x_0 is taken to be 0 (we assume the firm is beginning its production).

Our problem may then be stated as: for all nonnegative variables $x_1, \cdots, x_n, s_1, \cdots, s_n$ satisfying (2), where $r_1, r_2, \cdots, r_n$ are given positive constants, minimize

$$\sum s_t + \lambda \sum (x_t - x_{t-1})_+ , \tag{3}$$

where $1/\lambda$ units of increased production cost the same as 1 unit of storage. We solve this problem for all $\lambda \geqq 0$.

One method of attacking our problem is to write $x_t = y_{t-i} - z_t$, $y_t \geqq 0$, $z_t \geqq 0$, converting (3) to

$$\sum s_t + \lambda \sum y_t , \tag{3'}$$

and solve our problem using the simplex method. It is easy to prove that, for a solution, (3′) = (3). A method of solving linear programming problems, where the cost function involves a linear parameter, for all values of the parameter, has been developed by S. Gass, L. Goldstein, W. Jacobs and T. Saaty, and several such computations have been performed on SEAC, an electronic computer at the National Bureau of Standards in Washington. Our contribution here is (i) to describe some interesting properties of the solution to the problem, and (ii) to give a formula for the solution in the special case that the r_t are increasing.

3. *Theorem 1.* Let λ be specified. Then every solution to the problem has the property: if $x_{t+1} < x_t$, then $X_t = R_t$.

Proof: It is clear from (2) that a solution is specified if one specifies $x_1, \cdots, x_n$ (or, equivalently, $X_1, \cdots, X_n$). Assume the theorem false, and let t be the index exhibiting a violation of the property. Let $\epsilon > 0$ be less than $X_t - R_t$, x_t and $(x_t - x_{t+1})/2$. Let $x'_t = x_t - \epsilon$, $x'_{t+1} = x_{t+1} + \epsilon$, $x'_i = x_i$ for $i = 1, \cdots, n$, $i \neq t, t+1$. Then $(x'_1, \cdots, x'_n)$ satisfies the nonnegative requirements and (1). If we set $s'_i = X'_i - R_i$, then

$$\sum s'_i = \sum s_i - \epsilon < \sum s_i \tag{4}$$

Further,

$$(x'_t - x'_{t-1})_+ \leqq (x_t - x_{t-1})_+ , \tag{5}$$

since $x'_t < x_t$.

Also, since $x'_{t+1} - x'_t = x_{t+1} - x_t + 2\epsilon$, and $x_t - x_{t+1} > 2\epsilon$,

$$(x'_{t+1} - x'_t)_+ = (x_{t+1} - x_t)_+ = 0 \tag{6}$$

Finally, because $x'_{t+1} > x_{t+1}$, it follows that

$$(x'_{t+2} - x'_{t+1})_+ \leqq (x_{t+2} - x_{t+1})_+ . \tag{7}$$

For $i \neq t, t + 1, t + 2$, we have

$$(x'_i - x'_{i-1})_+ = (x'_i - x'_{i-1})_+ . \tag{8}$$

Thus, since $\lambda \geqq 0$, we have from (5), (6), (7), (8),

$$\lambda \sum (x'_i - x'_{i-1})_+ \leqq \lambda \sum (x_i - x_{i-1})_+ .$$

Combining this inequality with (4), we obtain a contradiction to the hypothesis that $(x_1, \cdots, x_n)$ is a solution.

Theorem 2. Let λ be specified. Then every solution to the problem satisfies $X_n = R_n$.

Proof: Assume the contrary. If $x_n > 0$, then it is easy to see, using the method of Theorem 1, that decreasing x_n slightly will yield a smaller value for (4). Assume then that $x_n = x_{n-1} = \cdots = x_{k+1} = 0$, $x_k > 0$. Then, by Theorem 1, $X_k = R_k$. Therefore $X_n = X_k < R_n$, since all r_i are positive. This is a contradiction to $X_n \geqq R_n$.

Theorem 3. Let K_t be the convex envelope of the function R_t (with $R_0 = 0$). Then for any prescribed value of λ, every solution $(X_1, \cdots, X_n)$ of the problem satisfies $R_t \leqq X_t \leqq K_t$.

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Proof: The left hand inequality is (1), so all that needs to be checked is $X_t \leqq K_t$. For ease in following the exposition, consider this diagram:

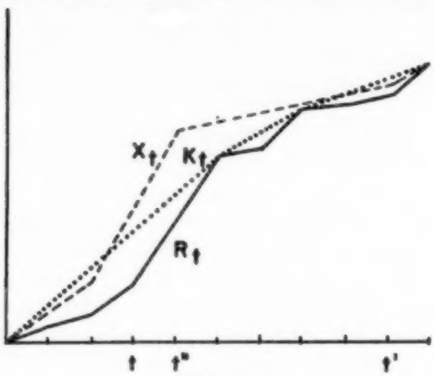

FIG. 1

Note that $x_{t+1} = X_{t+1} - X_t$ = slope of a chord of the polygonal curve X.

Now, assume that, for some t, $X_t > K_t$. Since, by Theorem 2, X_t eventually rejoins R_t, it follows that there is a smallest value of t, say t', such that $t' > t$, $X_{t'} \leqq K_{t'}$. Further, because $x_t >$ slope of K at $t >$ slope of K at $t' > x_{t'}$, there exists a t'', $t \leqq t'' < t'$ such that $x_{t''} > x_{t''+1}$. By theorem 1, this implies $X_{t''} = R_{t''}$, contradicting the definition of t'.

Corollary 1. For any value of λ, every solution X satisfies $X_t = R_t$ for each t such that $R_t = K_t$.

The significance of the corollary lies in the fact that it permits us to break up a possibly large problem into several smaller problems, namely from one value of t such that $R_t = K_t$ to the next. For it is easy to check that the variables

which appear in one such interval of values of t do not effectively appear in another. This saves substantial space, (and possibly time) in computation. This phenomenon, as the proof shows, depends only on the theorems, not on the particular economic model under discussion. All we ask of the model is that it satisfy (1) for the nonnegative variables $x_1, \cdots, x_n$ and Theorems 1 and 2.

4. We now turn to the special case $0 < r_1 < r_2 < \cdots < r_n$; i.e., R_t is a convex function of t. It follows from Theorem 1 that $0 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_n$; i.e., X_t is a convex function of t. For if $x_t > x_{t+1}$, then $X_t = R_t$ (Theorem 1), which implies $r_t \geqq x_t > x_{t+1} \geqq r_{t+1}$, a contradiction. Hence, our minimizing function (3) becomes

$$\sum s_i + \lambda x_n . \tag{9}$$

From here, we can proceed in one of two ways. Since (9) is a linear form defined on the closed, bounded, convex set Γ of all $(x_1, \cdots, x_n)$ satisfying the linear conditions (1), $X_n = R_n$, and $0 \leqq x_1 \leqq \cdots \leqq x_n$, we need only consider (9) at vertices of Γ, since a linear form defined on a convex polyhedron achieves its minimum at a vertex. Now the vertices of Γ are known [2] to be all X such that $X_t > R_t$ implies $x_t = x_{t+1}$; i.e., whenever X_t "leaves" R_t, it proceeds along an unbroken line. Examination of (9) clearly shows that the only vertices that have to be examined are those satisfying $x_1 = r_1, \cdots, x_k = r_k, x_{k+1} = x_{k+2} = \cdots = x_n = (R_n - R_k)/n - k$.

But we can reach the same point by an ad hoc discussion. Assume that $x_n = X_n - X_{n-1}$ has been chosen. How can (9) be minimized? What must be done, of course, is to complete X in such a way that $\sum X_i$ is minimal, since $\sum s_i = \sum X_i - \sum R_i$. Clearly, the minimal X has the following appearance:

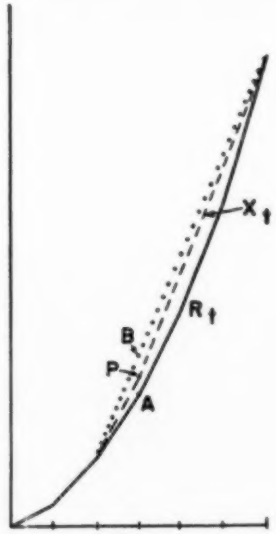

FIG. 2

Then it is easy to show that (9) depends linearly on the height of P as P varies between A and B. Hence, we may assume that P is A or B, since a linear function defined on an interval is either constant or takes its minimum at an end point. Thus $x_n = \dfrac{R_n - R_k}{n - k}$ for some k, $x_{k+1} = \cdots = x_n$, $x_1 = r_1, \cdots, x_k = r_k$, which is what we sought to establish.

Write

$$f(X) = \sum X_i + \lambda x_n .$$

Since $f(X)$ differs from (9) by a constant—namely $\sum R_i$—our task is to minimize $f(X)$. Denote by $X^{(k)}$ the function described in the preceding paragraph.

Theorem 4. If $\lambda \leqq \frac{1}{2}(n - k)(n - k + 1)$, then $f(X^{(k)}) \leqq f(X^{(k-1)})$. If $\lambda \geqq \frac{1}{2}(n - k)(n - k + 1)$, then $f(X^{(k)}) \geqq f(X^{(k-1)})$.

Proof:

$$f(X^{(k)}) = \sum_{i=1}^{k-2} R_i + \frac{R_n + R_k}{2}(n - k + 1) + \lambda \frac{R_n - R_k}{n - k}$$

Replace k by $k - 1$. Then

$$f(X^{(k)}) - f(X^{(k-1)}) = -\frac{1}{2}Y + \frac{\lambda Y}{(n-k)(n-k+1)},$$

where $Y = -(n - k)r_k + R_n - R_k > 0$. The conclusion of the theorem follows at once. Note that the range of k is $0, 1, \cdots, n - 1$.

Corollary 2. If $(n - k - 1)(n - k)/2 \leqq \lambda \leqq (n - k)(n - k + 1)/2$, then $X^{(k)}$ solves our problem, i.e., minimizes (9).

5. The methods used above may be extended to consider the following additional case: assume that the intersections of K_i with R_i are at $0 = t_0, t_1, \cdots, t_m = n$, and that from t_i to t_{i+1} the r_i increase. Then using the above methods one may show that the solution is as follows.

$$\frac{a(a-1)}{2} \leqq \lambda \leqq \frac{a(a+1)}{2}$$

(a a nonnegative integer), then

$$x_1 = r_1, \cdots, x_{t_1-a}, r_{t_1-a}, x_{t_1-a+1} = \cdots = x_{t_1} = \frac{R_{t_1} - R_{t_1-a}}{a}:$$

if $i \geqq 1$, and $t_{i+1} - t_i < 2a$, then

$$x_{t_i+1} = \cdots = x_{t_{i+1}} = \frac{R_{t_{i+1}} - R_{t_i}}{t_{i+1} - t_i};$$

if $i \geqq 1$, and $t_{i+1} - t_i \geqq 2a$, then

$$x_{t_i+1} = \cdots = x_{t_i+a}, \cdots, = \frac{R_{t_i+a} - R_{t_i}}{a},$$

$$x_{t_i+a+1} = r_{t_i+a+1}, \cdots, x_{t_{i+1}-a} = r_{t_{i+1}-a},$$

$$x_{t_{i+1}-a+1} = \cdots = x_{t_{i+1}} = \frac{R_{t_{i+1}} - R_{t_{i+1}-a}}{a}.$$

References

1. Charnes, A., Cooper, W. W., and Farr, D., "Linear Programming and Profit Preference Scheduling for a Manufacturing Firm," *Journal of the Operations Research Society of America*, May 1953, Vol. 1, no. 3, p. 114–129.
2. Hoffman, A., "On an Inequality of Hardy, Littlewood and Polya," an NBS report.
3. Magee, J. F., "Studies in Operations Research I: Application of Linear Programming to Production Scheduling," Arthur D. Little Inc., Cambridge, Mass.
4. Modigliani, F., and Hohn, F., "Solution of Certain Problems of Production Planning over Time Illustrating the Effect of the Inventory Constraint," Appendix to Cowles Commission Discussion Paper No. 2038.

A REMARK ON THE SMOOTHING PROBLEM*

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1. *Introduction:*

In a previous study [1] of the problem of smooth patterns of production—with which we presume the reader is familiar—a formula was found for the solution of the problem in the special case that the requirements r_i are increasing. In this note, we point out that a similar formula holds if $0 < r_1 < \cdots < r_{n-1}$, $r_{n-1} > r_n > 0$. Actually, as will be clear from our method, the formula holds more generally, namely, if the r_i increase up to a certain value i_0, then decrease to $r_n > 0$. For simplicity, however, we confine our attention to the case $i_0 = n - 1$.

2. *The problem:*

For each $\lambda \geqq 0$, we must minimize

$$\sum_{t=1}^{n} X_t + \lambda \sum_{t=1}^{n} (x_t - x_{t-1})_+, \tag{1}$$

where

$$X_t = \sum_{i=1}^{t} x_i, \qquad x_i \geqq 0, \quad x_0 = 0;$$

$$x_t \geqq R_t, \qquad t = 1, \cdots, n, \tag{2}$$

where

$$R_t = \sum_{i=1}^{t} r_i, \text{ and}$$

$$0 < r_1 < \cdots < r_{n-1}\,;\, r_{n-1} > r_n > 0. \tag{3}$$

From [1] we inherit the following facts:

(a) We may assume $R_n/n < r_n$ (otherwise there is no problem).

$$X_n = R_n \tag{b}$$

(c) $r_i < r_{i+1}$ implies $x_i \leqq x_{i+1}$ for any minimal solution $x_1, \cdots, x_n$.

(d) If $x_{i+1} < x_i$ for some minimal solution, then $X_i = R_i$. In particular, $x_n < x_{n-1}$ implies $X_{n-1} = R_{n-1}$, $x_n = r_n$.

Let us now consider the set of all minimal solutions satisfying $x_{n-1} > x_n$. Then the coefficient of λ in (1) becomes, by (c), x_{n-1}. If x_{n-1} is specified, then clearly the optimal way to complete the solution is in such a way that the polygonal curve X_i is as "low" as possible subject to (2), (b), (c) and the specified x_{n-1} and x_n. Now either there is some i, $1 < i \leqq n - 2$ such that $(R_{n-1} - R_i)/$

* This work was supported (in part) by the Office of Scientific Research, USAF.

$(n - 1 - i) = x_{n-1}$ or not. If not, it is easy to see that the "low" curve previously specified is the midpoint of two other curves, each satisfying $x_{n-1} < x_n$ and the given conditions, and thus may be ignored if we are making up a minimal list of candidates for the minimum of (1). (Note that once we require $x_{n-1} < x_n$, (1) has become a linear function). Thus, if we wish to consider the minimal solutions satisfying $x_{n-1} < x_n$, we may confine our attention to the following (which we call the *concave* class):

Let k be the largest index satisfying

$$\frac{R_n - R_k}{n - k} \leqq r_n. \tag{4}$$

(Note $k < n - 2$, since $r_n < r_{n-1}$.) Then the members of the concave class are all $x_1, \cdots, x_n$ satisfying

$$\begin{aligned} x_i &= r_i, \qquad i = 1, \cdots, l, \text{ where } k < l < n - 1, \\ x_{l+1} = \cdots = x_{n-1} &= \frac{R_{n-1} - R_l}{(n - l - 1)}, \\ x_n &= r_n. \end{aligned} \tag{5}$$

Similarly, let us consider all solutions satisfying[1] $x_{n-1} < x_n$, so that the coefficient of λ in (1) becomes x_n. By reasoning similar to the above, we may confine our attention to the *convex* class:

$$\begin{aligned} x_i &= r_i, \qquad i = 1, \cdots, l, \text{ where } l \leqq k, \\ x_{l+1} = \cdots = x_n &= \frac{R_n - R_l}{n - l}. \end{aligned} \tag{6}$$

If we have equality in (4), then we have specified all solutions. If not, reasoning similar to the above requires the specification of an additional solution, called *exceptional*:

$$\begin{aligned} x_i &= r_i, \qquad i = 1, \cdots, k, \\ X_{k+1} &= (n - k - 1)R_{n-1} - (n - k - 2)R_n, \\ x_{k+2} &= \cdots = x_n = r_n. \end{aligned} \tag{7}$$

In what follows, we shall assume that equality has not held in (4). If it has, all the remarks to be made about the exceptional solution apply to the convex with $l = k$.

Let us denote by $f_l(\lambda)$ the value of (1) for $l = 1, \cdots, k$ (convex class); $k + 1, \cdots, n$ (concave class). Denote by $f_e(\lambda)$ the value of (1) for the exceptional solution. We inherit from [1] the information:

(e) $\lambda \geqq (n - k - 2)(n - k - 1)/2$ implies that $f_{k+1}(\lambda)$ is the optimal solution among members of the concave class.

[1] We assume here that we do *not* have equality in (4). If equality does hold, then we consider here all solutions satisfying $x_{n-1} \leqq x_n$.

(f) $\lambda \leqq (n-k)(n-k+1)/2$ implies that $f_k(\lambda)$ is the optimal solution among members of the convex class.

(e′) For any λ, a formula for the optimal among the concave solutions.

(f′) For any λ, a formula for the optimal among the convex solutions.

Our problem then is to choose among the exceptional solution and the concave and convex solutions, the optimal for a given value of λ.

3. *The result:*

For $\lambda \leqq (n-k-1)(n-k-2)/2$, the optimal solution is *the optimal concave solution. For $(n-k-1)(n-k-2)/2 \leqq \lambda \leqq (n-k-1)(n-k)/2$, the optimal solution is the exceptional solution. For $\lambda \geqq (n-k-1)(n-k)/2$, the optimal solution is the optimal convex solution.*

Proof: Evaluating (1), we obtain

$$(8)\quad f_{k+1}(\lambda) = \sum_{i=1}^{k} R_i + \frac{1}{2}(n-k-1)(R_{k+1} + R_{n-1}) + R_n + \lambda \frac{R_{n-1} - R_{k+1}}{n-k-2},$$

$$(9)\quad f_e(\lambda) = \sum_{i=1}^{k} R_i + \frac{n-k}{2}[(n-k-1)R_{n-1} - (n-k-3)R_n] + \lambda(R_n - R_{n-1}),$$

$$(10)\quad f_k(\lambda) = \sum_{i=1}^{k-1} R_i + \frac{1}{2}(n-k+1)(R_k + R_n) + \lambda \frac{R_n - R_k}{n-k}.$$

It is easy to compute from (8), (9) and (10) that

$$(11)\quad f_{k+1}((n-k-1)(n-k-2)/2) = f_e((n-k-1)(n-k-2)/2)$$

and

$$(12)\quad f_k((n-k)(n-k-1)/2) = f_e((n-k)(n-k-1)/2).$$

But $f_{k+1}(\lambda) - f_e(\lambda)$ is a linear function of λ, and obviously $f_{k+1}(0) - f_e(0) \leqq 0$. Therefore, $\lambda \leqq (n-k-1)(n-k-2)/2$ and (11) imply

$$(13)\quad f_{k+1}(\lambda) \leqq f_e(\lambda).$$

Similarly, (12) and $f_k(0) - f_e(0) \geqq 0$ imply that if $\lambda \leqq (n-k)(n-k-1)/2$, then

$$(14)\quad f_e(\lambda) \leqq f_k(\lambda).$$

Since (14) holds, a fortiori, for $\lambda \leqq (n-k-1)(n-k-2)/2$, it follows from (13), (14), (f) and (e′) that for λ in this range, the optimal solution is the optimal concave solution, and can be determined.

On the other hand, let

$$\lambda \geqq (n-k-1)(n-k)/2$$

Then from (11), we have

$$(15)\quad f_{k+1}(\lambda) \geqq f_e(\lambda),$$

and from (12), we obtain

$$f_e(\lambda) \geqq f_k(\lambda). \tag{16}$$

Then (15), (16), (e) and (f′) imply that for λ in this range, the optimal solution is the optimal convex solution, and can be determined.

Finally, suppose

$$(n-k-1)(n-k-2)/2 \leqq \lambda \leqq (n-k)(n-k-1)/2. \tag{17}$$

For λ in this range, $f_{k+1}(\lambda)$ is the optimal concave solution, $f_k(\lambda)$ is the optimal convex solution, by (e) and (f); and (11) and (12) imply that in this range, $f_e(\lambda) \leqq f_k(\lambda)$, $f_e(\lambda) \leqq f_{k+1}(\lambda)$. Hence the exceptional solution is the optimal solution.

Reference

1. Hoffman, A. J. and Jacobs, W., "Smooth Patterns of Production," this issue, pp. 86–91.

POLICY STATEMENT FOR *MANAGEMENT SCIENCE*

MANAGEMENT SCIENCE is a journal, the objective of which is to identify, extend, and unify scientific knowledge that contributes to the understanding and practice of management.

Articles in MANAGEMENT SCIENCE are directed to all persons interested in developing or applying scientific knowledge which is relevant to the field of management.

The journal will publish the following types of article:

Evaluative articles which are designed to portray, explain and assess developments in fields which are relevant to the management sciences. These articles will be expressed in non-technical language so that they will be readily accessible to persons in managerial positions.

Method articles which are designed to explain the methods that scientific research has made available for various kinds of management problems. These articles will be expressed in a language which is readily accessible to engineers and scientists who are interested in applying these methods to practical problems.

Survey articles which are designed to assess selected aspects of various fields which are or can be made relevant to the management sciences. These articles are intended to provide guidance for research workers in the management sciences. As such they may be expressed in technical scientific language modified in a manner which will make them suitable for persons who do not possess backgrounds of specialization in the fields being reviewed.

Technical articles which are designed to present the results of original research in theory or application. These articles will be directed toward persons who are intimately concerned with such research. Technical terminology may be utilized in these articles to whatever extent is necessary to convey the nature and significance of the work undertaken and the results that have been achieved.

Articles from any field (industrial engineering, econometrics, operations research, mathematics, statistics, psychology, sociology, political science, etc.) will be welcomed as long as the article furthers the aim of developing a unified science of management. Thus the manner of writing of an article will be important in guiding the Board of Editors. An article written chiefly from the point of view of industrial engineering, for example, would not be acceptable, but an article on the same topic might properly be accepted if written from the point of view of the development of management science.

C. West Churchman
Managing Editor

CONSTITUTION AND BY-LAWS

of

THE INSTITUTE OF MANAGEMENT SCIENCES

Constitution

(Adopted at the Founding Meeting of December 1, 1953.)

Article I. Name

The name of this organization shall be The Institute of Management Sciences.

Article II. Objects

The objects of the Institute shall be to identify, extend, and unify scientific knowledge that contributes to the understanding and practice of management. To this end, the Institute proposes to conduct meetings; to produce publications devoted to management science and its applications; to cooperate with other organizations in the advancement of the practice of management; to stimulate research and promote high professional standards in the development of a unified management science; and in general, to promote the growth of management science and its practice.

Article III. Membership

1. *Classes of Membership*

The membership of the Institute shall comprise Individual Members and Institutional Members. Except as otherwise provided by the Constitution, the right to vote, to sign referendum petitions and the like, to hold office, and to sign nominating petitions shall be limited to Individual Members.

2. *Qualifications for Membership*

Membership shall be open to all individuals and institutions who are interested in promoting the growth of management science and its practice and who meet the qualifications set forth in the By-Laws.

Article IV. Officers

1. *Officers*

The Officers of the Institute shall be a Past President, President, President-Elect, three Vice-Presidents, Past Secretary, Secretary-Treasurer, and Associate Secretary.

2. *Terms of Office*

The President-Elect shall serve for one year and at the close of this term automatically become President; upon completing one year as President, he shall serve one year as Past President. The Associate Secretary shall serve for one year and at the close of this term automatically become Secretary-Treasurer;

upon completing one year as Secretary-Treasurer, he shall serve one year as Past Secretary. The Vice-Presidents shall be elected for a three-year term, one Vice-President being elected each year. Terms of office shall begin on January 1, but each Officer shall serve until his successor takes office.

3. *Method of Election*

On or before June 15, of each year, the Committee on Elections shall submit through the Secretary-Treasurer one nomination for President-Elect, two nominations for Vice President, one nomination for Associate Secretary, one nomination for President if the office of President-Elect is vacated before May 15, one nomination for Secretary-Treasurer if the office of Associate Secretary is vacated before May 15, and two nominations for each other office vacated before May 15. These nominations shall be published in the next *Bulletin*. Additional nominations may be made within five weeks after this publication, by petition signed by at least twenty-five Individual Members and submitted to the Secretary-Treasurer.

On or before November 15 the Secretary-Treasurer shall mail to all Individual Members a brief biographical sketch of each nominee for office, with a preferential ballot for the election of officers from among the Members so nominated. The Committee on Elections shall determine the election from a count of ballots returned to the Secretary-Treasurer by December 15.

4. *Vacancies in Office*

The Council shall fill any vacancy which may occur between elections in any office of the Institute except those otherwise provided for in the Constitution.

If a vacancy occurs in the office of President, the Vice-President with longest service in office shall become President.

If a vacancy occurs in the office of Past President, the member holding this office most recently in the past shall become Past President.

5. *Duties*

The Past President shall preside over Council meetings, serve as Chairman of the Council, and appoint Committees voted by the Council.

The President shall serve as a member of the Council, be the chief executive officer of the Institute, and plan and administer the affairs of the Institute.

The President-Elect shall serve as a member of the Council.

The Vice-Presidents shall serve as members of the Council and shall advise and assist the President in his executive capacity. In the event that the President is absent or unable to serve, one of the Vice-Presidents, selected in order of length of service in that office, shall act as President.

The Past Secretary shall serve as Vice-Chairman of the Council, and shall act as Chairman in the event that the Past President is temporarily absent or unable to serve.

The Secretary-Treasurer shall serve as Recording Secretary of the Council, keep and publish the By-Laws, serve as Chairman of the Committee on Elections, maintain the membership roll, receive and disburse Institute funds, and keep such other records and perform such other duties as are specified in the Constitution and By-Laws.

The Associate Secretary shall serve as a member of the Council, and shall advise and assist the Secretary-Treasurer in the performance of his duties.

Article V. Government

1. *Council*

The Institute shall be governed by a Council composed of the nine Officers specified in Article IV. The Past President shall serve as Chairman, the Past Secretary as Vice-Chairman, and the Secretary-Treasurer as Recording Secretary of the Council.

As the policy-making body of the Institute, The Council shall establish broad programs and policies for the conduct of Institute affairs by making suitable provisions in the By-Laws.

The Council shall adopt and amend the By-Laws so as to further the objects of the Institute. Each such amendment shall be published in the Bulletin and shall not become effective until at least thirty days have elapsed after distribution of the issue of the Bulletin in which it is announced.

The Council shall fill each interim vacancy in its own membership by a temporary appointment terminating when the vacancy is filled by election under the provisions of Article IV, Section 3.

The Council shall establish a Committee on Elections before May 15, including the Secretary-Treasurer as Chairman-ex-officio and four other Individual Members of whom two have not, and two have, served as Officers of the Institute.

The Council may establish standing committees, but no individual may be appointed to serve for a period longer than three years and not more than two-thirds of the members of a standing committee may serve in any two consecutive years.

At least five affirmative votes shall be necessary and sufficient for each action by the Council. The Council may be polled either by mail or in a meeting, but votes by proxy shall not be counted.

2. *President*

As chief executive officer of the Institute, the President shall be responsible to the Council for conducting the Institute's affairs so as best to further its objects in a manner consistent with the Constitution and By-Laws.

The President shall be the chief representative of the Institute before the public and in relationships of the Institute with other persons and organizations.

3. *Secretary-Treasurer*

The Secretary-Treasurer shall be responsible to the Council for an accurate and complete reporting of the financial affairs of the Institute, for safe-keeping of funds, for receipt and disbursement of funds in accordance with provisions of the Constitution and By-Laws, and for financial operations otherwise compatible with the plans and directives of the President acting in his executive capacity.

The Secretary-Treasurer shall provide the Council with such reports as it may require on membership, meetings, publications, and other activities of the Institute.

The Secretary-Treasurer shall make all Institute records in his custody avail-

able also for use by the President, except as the Council may otherwise require.

The Secretary-Treasurer shall render to the membership annually a full report of his stewardship of the Institute's resources, such report to be audited by a Finance Committee appointed for that purpose by the President with the approval by the Council.

4. *Referendum*

Upon petition of 15 Individual Members, a proposal for amendment of the By-Laws shall be published in the Bulletin and brought before the Council. Upon written request of 100 Individual Members, the membership shall be polled by mail ballot upon any proposal either rescinding past action of the Council or President or proposing new action, in which case the action of the membership shall govern. These requests shall be transmitted to the Secretary-Treasurer.

Article VI. Meetings

1. *Institute*

There shall be at least one general meeting of the Institute each year, for the purpose of discussing the management sciences and their applications to the practice of management.

2. *Council*

There shall be at least one meeting of the Council each year. Meetings of the Council shall be held whenever called by the Past President, or requested by six members of the Council.

Article VII. Publications

1. *Journal*

To provide a medium for disseminating knowledge of the management sciences, to stimulate research in these sciences, and to encourage and improve applications of this knowledge the Institute shall undertake to publish a Journal, to be called *Management Science*.

2. *Bulletin*

The Institute shall publish a *Bulletin of the Institute of Management Sciences*, which shall be a periodical distributed to all Individual Members as a privilege of membership. The Bulletin shall constitute the official medium of the Institute for the publication of announcements and reports, notices of meetings, elections and appointments, and such other items as are required or authorized by the Constitution and By-Laws.

3. *Publication Management*

The Council shall establish policies governing the publication program, including the *Journal*, the *Bulletin*, and other periodical and nonperiodical publications of the Institute.

The Council may appoint the Editors of any or all Institute publications, may establish Editorial Boards, and may delegate power to appoint Editors and Editorial Boards to the President or to standing committees established by the Council.

Article VIII. Honors

1. *Authority*

The Institute, acting through the Council, may bestow upon any individual or group such honors or recognition as the Council deems will serve to advance the management sciences.

2. *Qualifications*

No Officer may be accorded such honors or recognition during his term of office or for two years thereafter. No qualification other than merit as evidenced by a record of past accomplishments in advancing knowledge of the management sciences or their application shall serve as a basis for such recognition.

Article IX. Amendments

1. *Proposal*

Amendments to the Constitution may be proposed by the Council or by a petition signed by 25 Individual Members. An amendment originating by petition shall be referred to the Council for its recommendation as to ratification.

2. *Ratification*

Following action by the Council, the Secretary-Treasurer shall publish a copy of the proposed amendment and the Board's recommendation in the next issue of the *Bulletin*, inviting comment. Comments received from Individual Members shall be summarized or published in full in subsequent issues of the *Bulletin*. Unless the Council decides that the proposed amendment is of such urgency as to require a special mail ballot, the amendment shall be submitted to the Individual Members for mail vote at the time of the annual election of Officers. Ratification shall require an affirmative vote of two-thirds of the mail ballot, counting to be done by the Committee on Elections.

By-Laws

Article I. Dues

1. The annual dues of Individual Members shall be ten dollars, except that the annual dues of individual members residing outside the boundaries of North America shall be five dollars.
2. The annual dues of Institutional Members shall be one hundred dollars.
3. The annual period for payment of dues shall be the fiscal period extending from June 30 to the same date in the next succeeding calendar year.
4. Each Institutional Member shall have the privilege of naming as representative one Individual Member, who shall not then be liable for dues.
5. The Council may, in its discretion, waive the payment of part or all of the dues of an individual member during any annual period.

Article II. Membership

1. Candidates for Individual Members shall be nominated by two Members, by writing to the Secretary-Treasurer, who shall enroll the new Member.
2. Any Individual or Institutional Member may resign upon written notice to the Secretary-Treasurer.

3. Nonpayment of dues beyond six months after the close of the fiscal year shall cause the removal of the Member's name from the membership list.

4. Any Member may be removed from membership by a majority of the Members, in attendance at a general meeting of the Institute, for conduct deemed prejudicial to the Institute, provided that such Member shall have first been served with written notice of the accusation against him and shall have been given an opportunity to produce his witnesses, if any, and to be heard in the meeting at which such vote is taken.

Article III. The Council

1. A meeting of six members of the Council, properly convened, shall constitute a quorum.

2. The Past President shall have the power to convene the Council as he deems necessary, with a minimum of one meeting a year.

3. Upon petition by six members of the Council the Past President shall, also, convene the Council at such time and place and for such purposes as shall be designated in that petition, except that such petition shall not require less than 30 days notice from Past President to the Council.

Article IV. Publications

1. Each fully paid Individual Member shall receive one copy of the *Bulletin*.

2. The *Bulletin* and *Journal* may be combined into one periodical.

3. The Editorial Board of the Journal, appointed by the Council, shall be a standing committee of the Institute. The Board shall consist of six Individual Members appointed by the President, with the approval of the Council, each for a term of three years with two new members appointed each year. The Board shall be responsible for establishing and maintaining an editorial and operating policy for the *Journal* consistent with the provisions of Constitution and By-Laws. The Board shall make nominations to the Council for all appointments of Editors and Members of the Editorial Board.

Article V. Rules of Order

The rules contained in Roberts Rules of Order shall govern the Institute in all cases to which they are applicable, and in which they are not inconsistent with the By-Laws or the special rules of order of this Institute.

THE INSTITUTE OF MANAGEMENT SCIENCES

Application for Individual Membership

TO THE COUNCIL:

GENTLEMEN:

I hereby make application for membership in The Institute of Management Sciences on the basis of my interest in promoting the practice and growth of management sciences.

____________________ DATE

____________________ SIGNATURE

Remittance Enclosed $__________

(Individual membership dues are $10 per year*.) Make your check or money order payable to The Institute of Management Sciences and mail to George Kozmetsky, Secy-Treasurer, Litton Industries, 336 N. Foothill Road, Beverly Hills, California.

Do Not Use Space Below

Receipt Acknowledged__________

Elected______No__________

Notified__________

Names of References:

Please supply the names of two members for reference. Signatures not necessary; PRINT.

1)____________________ NAME OF REFERENCE

____________________ ADDRESS

2)____________________ NAME OF REFERENCE

____________________ ADDRESS

ADDRESS FOR MANAGEMENT SCIENCE AND OTHER MAIL

____

FIRST NAME MIDDLE INITIAL FAMILY NAME

____

MAILING ADDRESS

____

CITY POSTAL ZONE STATE

$8.00 represents payment for subscription to MANAGEMENT SCIENCE. The remaining $2.00 covers other services provided by The Institute of Management Sciences.

MANAGEMENT SCIENCE

Manuscripts intended for publication may be sent to any one of the Editors or to the Managing Editor. Manuscripts should be typewritten, double-spaced, on white 8½"x11" paper and should not exceed 30 pages. The general style of the journal should be followed in listing the title, author, etc., as well as in typing the references. Footnotes should be numbered consecutively and should appear on a separate sheet at the end of the paper. Figures should be drawn in black ink on white paper in a form suitable for photographic reproduction and, if necessary, reduction.

The Institute of Management Sciences does not necessarily subscribe in full to the theories, methods and opinions published in Management Science.

Publications intended for review and all editorial correspondence should be sent to the Managing Editor, C. West Churchman, Case Institute of Technology, Cleveland 6, Ohio.

Subscriptions, advertising and all business correspondence should be sent to the Business Manager, Alan O. Mann, SKF Industries, Inc., Box 6731, Philadelphia 32, Pennsylvania. *Subscription rate:* \$2.50 per issue; \$10.00 per year (\$10.50 outside U.S. and Canada).

Information concerning The Institute of Management Sciences and membership therein may be obtained from the Secretary-Treasurer, George Kozmetsky, Litton Industries, 336 N. Foothill Rd., Beverly Hills, California.